Antidifferentiation (or integration) of functions (fr : Primitives)

In many problems in calculus, we know the rate of change of one variable with respect to another, but we do not have a formula which relates the variables. In other words, we know $f'(x)$ but we need to know $f(x)$.

The process of finding $f(x)$ from $f'(x)$ is the reverse process of differentiation, it is called antidifferentiation.

1. ANTIDERIVATIVE OF A FUNCTION

Definition :
Let f be a differentiable function on an interval I. We say that F is an antiderivative (or primitive) of f on I if F is differentiable on I and if $F'(x)=f(x)$ for any x in I.

Example : 1. $F(x)=\frac{2}{x}$ is an antiderivative on $\mathbb{R}$ of the function defined by $f(x)=-\frac{2}{x^2}$.

Note : $x\mapsto\frac{2}{x}-5$ is a primitive of f as well.

2. $F(x)=x^4-\frac{x^2}{2}+5x-2$ is an antiderivative on $\mathbb{R}$ of the function defined by $f(x)=$…….

Property 1 :
Let f be a differentiable function on an interval I and F one antiderivative of f on I. Then all the primitives of f on I are the functions in the form $x\mapsto F(x)+k$ with $k\in\mathbb{R}$.

Proof : * let G be a such function : for any x in I, $G(x)=F(x)+k$. We thus have $G'(x)=F'(x)+0=F'(x)=f(x)$ so G is a primitive of f on I.

* conversely, let H be a primitive of f on I, we then have $H'(x)=f(x)=F'(x)$ for any x in I, so $H'(x)-F'(x)=0=(H-F)'(x)$, and so $H-F$ is constant on I. Is value is a real number k. Now $(H-F)(x)=k$, and finally $H(x)=F(x)+k$

Property 2:
Let f be a function defined on an interval I and $x_0\in I$. For any number y_0, there exists one and only one primitive F_0 of f on I such that $F(x_0)=y_0$.

Proof : Obvious consequence of the previous property.

Example : $f(x)=2x-1$ defined on $\mathbb{R}$. Let's find the unique primitive of f whose value is $y_0=5$ at $x_0=-2$: The primitives are in the form $F(x)=x^2-x+k$. $F(-2)=5\Leftrightarrow(-2)^2+2+k=5\Leftrightarrow k=5-6=-1$. This unique primitive is defined by $F(x)=x^2-x-1$

2. PRIMITIVES OF STANDARD FUNCTIONS

We get the table of standard functions' primitives by reading the table of derivatives backwards.

Function f	**Primitives F ,** $k \in \mathbb{R}$	**Interval I**
$f(x) = a$	$F(x) = ax + k$	$\mathbb{R}$
$f(x) = x^n$ $n \geq 1$	$F(x) = \dfrac{x^{n+1}}{n+1} + k$	$\mathbb{R}$
$f(x) = x^n$ $n \leq -2$	$F(x) = \dfrac{x^{n+1}}{n+1} + k$	$]0;+\infty[$ or $]-\infty;0[$
$f(x) = \dfrac{1}{x^2}$	$F(x) = \dfrac{-1}{x} + k$	$]0;+\infty[$ or $]-\infty;0[$
$f(x) = \dfrac{1}{\sqrt{x}}$	$F(x) = 2\sqrt{x} + k$	$]0;+\infty[$
$f(x) = e^x$	$F(x) = e^x + k$	$\mathbb{R}$

And the same way the table of operations on primitives : U et V being primitives of u and v on I

Function f	**Primitives F ,** $k \in \mathbb{R}$	**Interval**
$f = \alpha u' + \beta v'$	$F = \alpha u + \beta v + k$	I
$f = u' \times v \circ u$	$F = v \circ u + k$	I
$f = u' \times u^n$ $n \geq 1$	$F = \dfrac{u^{n+1}}{n+1} + k$	I
$f = u' \times u^n$ $n \leq -2$	$F = \dfrac{u^{n+1}}{n+1} + k$	$u \neq 0$ on I
$f = \dfrac{u'}{u^2}$	$F = \dfrac{-1}{u} + k$	$u \neq 0$ on I
$f = \dfrac{u'}{\sqrt{u}}$	$F = 2\sqrt{u} + k$	$u > 0$ on I
$f = u' e^u$	$F = e^u + k$	I

Key points of the chapter

- ✓ Knowing what is an antiderivative of a function
- ✓ Being able to find antiderivatives for usual functions