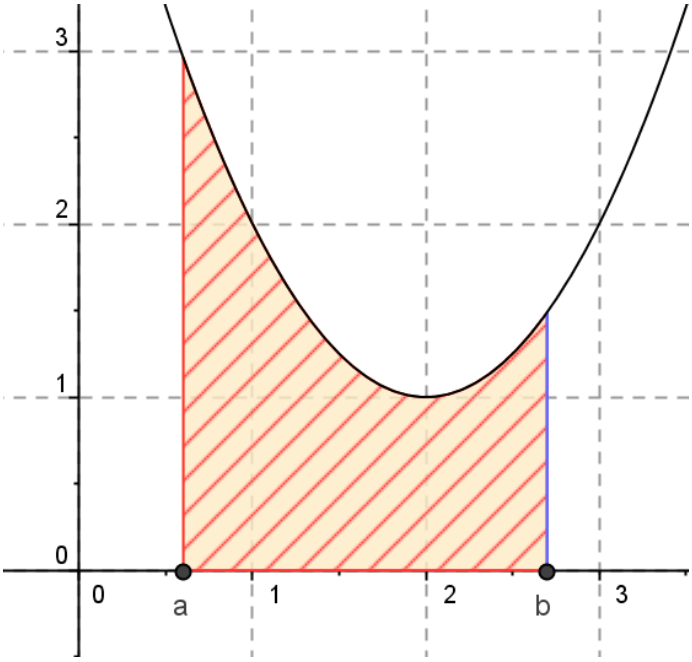
INTEGRATION

1. INTEGRAL OF A POSITIVE FUNCTION

We call *unit of area (fr : unité d'aire)* (denoted UA) of an orthogonal coordinate system $(O;\vec{i},\vec{j})$ the area of the rectangle built on the vectors $\vec{i}$ and $\vec{j}$:

Definition :
Let's consider f a continuous and positive function on an interval $[a;b]$ and C its graph. The area (measured in UA) of the domain D bounded by, the x-axix and the lines with equation $x=a$ and $x=b$ is a real number denoted $\int_a^b f(x)dx$ (which reads « integral from a to b of f »). x is a silent variable, a and b are the bounds of the integral.

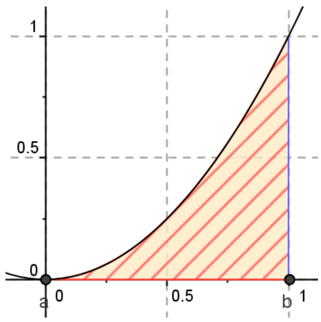


<u>Examples</u>

1. $f(x)=5$ (don't worry, we will see pretty soon less simple functions !).

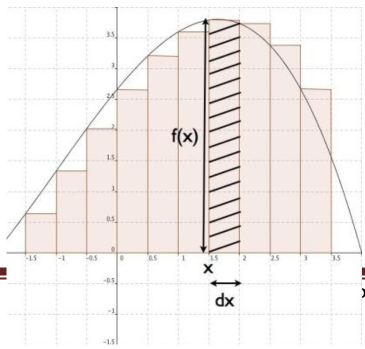
$\int_{-2}^{4} f(x)dx = 30$ UA , the domain D being then a rectangle whose sides are 6 and 5.

2. We will see that $\int_0^1 x^2 dx = \frac{1}{3}$



<u>Note :</u>

The meaning of the notation is « sum (hence the shape of the symbol $\int$) of the areas of the rectangles with height $f(x)$ and width dx (and thus having an area of $f(x)dx$) between a and b.



Definition (mean value) :
Let's consider f a continuous and positive function on an interval $[a;b]$. Its mean value on $[a;b]$ is $\mu = \frac{1}{b-a}\int_a^b f(x)\,dx$

Note :

1. We thus have $\mu(b-a) = \int_a^b f(x)\,dx$. The mean value μ is thus such that the area of the domain D under the graph is the same that the one of the rectangle with width $b-a$ and height μ.
2. If x is express in a unit A and y (or $f(x)$) in a unit B, then $\int_a^b f(x)\,dx$ is homogenous to $A \times B$ and μ to B.

Example :

Since $\int_0^1 x^2 dx = \frac{1}{3}$, the mean value of the cube function between 0 and 1 is $\mu = \frac{1}{1-0}\int_0^1 x^2 dx = \frac{1}{3}$

2. Properties

The following properties will be admitted.

Properties:

(positivity) : If $f(x) \geq 0$ ***on*** $[a;b]$, ***then*** $\int_a^b f(x)\,dx \geq 0$

(linearity) : For any real numbers α ***and*** β ***and any continuous functions f and g on*** $[a;b]$ ***we have :***
$$\int_a^b (\alpha f(x) + \beta g(x))\,dx = \alpha\int_a^b f(x)\,dx + \beta\int_a^b g(x)\,dx$$

(Chasles's relation) : For any real number c in $[a;b]$ ***we have :***
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

(order) : If $f(x) \leq g(x)$ ***on*** $[a;b]$, ***then :*** $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$

Example :

3. CALCULATING AREAS (LINK WITH PRIMITIVES)

- ✓ Sir Isaac Newton and Gottfried Leibniz showed at the end of the XVII[th] century the link between differential calculus and the integral. This link is called the Fundamental Theorem of Calculus. The beauty and power of this theorem is that it enables us to evaluate complicated summations and areas.
- ✓ See introducing TP.

> ***Fundamental Theorem of Calculus :***
> ***Let's consider f a continuous, increasing and positive function on an interval $[a;b]$ and a a real number in I. There exists a unique primitive of f such that $F(a)=0$ and this primitive is defined on I by*** $F(x)=\int_a^x f(t)\,dt$

> ***Property :***
> ***f being a continuous function on an interval I and F a (indifferent) primitive of f on I,***
> $\int_a^b f(t)\,dt = F(b)-F(a)$

Proof :

Let's denote G the primitive of f on I such that $G(a)=0$ (whose existence is given by the fundamental theorem) : $G(x)=\int_a^x f(t)\,dt$. We know that $F=G+k$ with k real number, and then $F(b)-F(a)=G(b)+k-(G(a)+k)=G(b)-G(a)=\int_a^b f(t)\,dt$

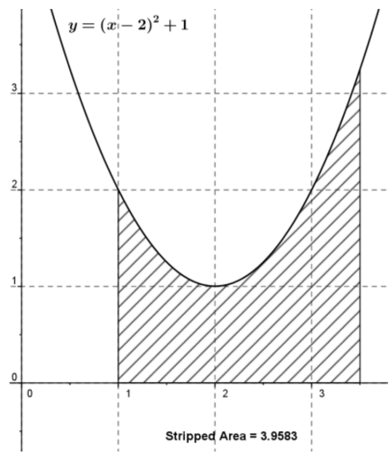
Note :

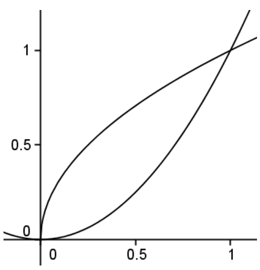
We often use the notation $\int_a^b f(t)\,dt = [F(t)]_a^b$ which avoids writing explicitly the primitive :

$\int_a^b (5x-e^x)\,dx = \left[\frac{5}{2}x^2-e^x\right]_a^b = \dots$

Example :

1. Calculate (in UA) the stripped area.





2. Calculate the area of the domain between the graph of the functions $x \mapsto \sqrt{x}$ and $x \mapsto x^2$.