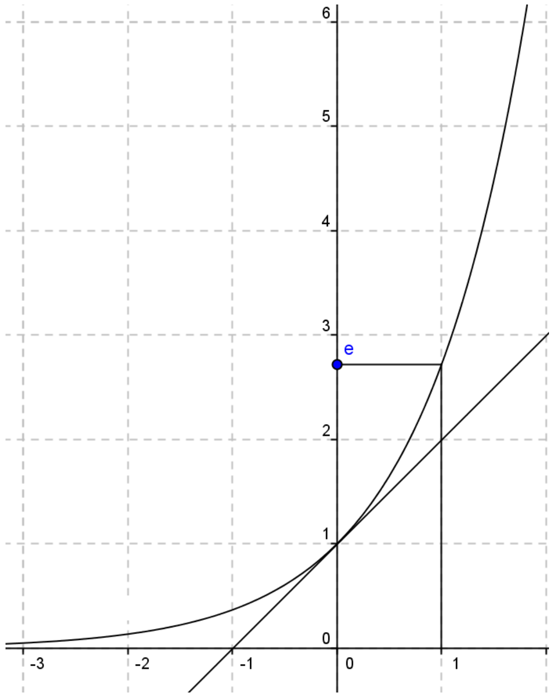
The Neperian (or natural) logarithm function
Fr : logarithme népérien

1. Definition and first properties



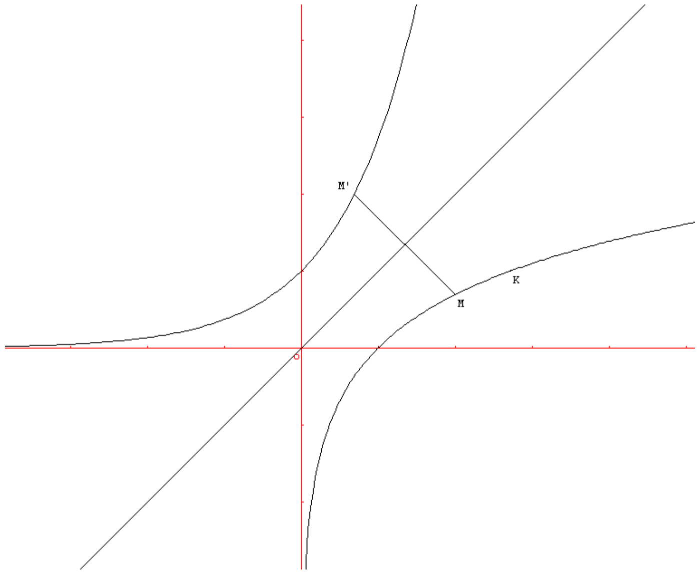
The exponential function $y \mapsto$ is continuous and strictly increasing on $\mathbb{R}$. For any x in $\mathbb{R}^{+*}$, there thus exists a unique (thanks to the intermediate value theorem) real number y such that $e^y = x$.

Definition :
The natural logarithm function (fr : fonction logarithme népérien[1]), denoted ln is defined on $]0;+\infty[$ ***by :***
$x \mapsto \quad x)$ ***with*** $e^y = x$

Note :

- we denote $\ln(x)$ or $\ln x$
- ***ln*** key on calculators : $\ln 5 \approx$ $\ln 100 \approx$

Properties :
1. For any $x > 0$ **,** $e^y = x \Leftrightarrow y = \ln x$ ***we say that the functions exponential et natural logarithm are <u>inverse</u> (fr <u>réciproques</u>)***
2. Their graph are <u>symmetrical about the line with equation</u> $y = x$
3. For any $x > 0$ **,** $e^{\ln x} = x$
4. For any $x \in \mathbb{R}$ **,** $\ln e^x = x$
5. $\ln 1 = 0$ ***and*** $\ln e = 1$



Proof (2) :
$M(x;y) \in C_{\ln} \Leftrightarrow y = \ln x \Leftrightarrow x = e^y \Leftrightarrow M'(y;x) \in C_{\exp}$ M' is symmetrical to M about the line with equation $y = x$

2. Study of the ln function

2.1 Caracteristic property.

For any strictly positive real numbers a and b : $\ln(ab) = \ln(a) + \ln(b)$.

[1] Named after John Napier of Merchiston (1550 – 4 April 1617) – also signed as Neper – named Marvellous Merchiston, was a Scottish mathematician, physicist, astronomer & astrologer, who as invented them

2.2 CALCULATIONS PROPERTIES.

For any strictly positive real numbers a and b and n integer number:

$\ln\left(\frac{1}{a}\right) = -\ln a$ $\qquad \ln\left(\frac{a}{b}\right) = \ln a - \ln b$ $\qquad \ln\left(a^n\right) = n\ln a$ $\qquad \ln\left(\sqrt{a}\right) = \frac{1}{2}\ln a$

proofs : direct consequences of the previous property :

2.3 CONTINUITY AND DIFFERENTIABILITY.

Property : on $]0;+\infty[$ ***,***

1. The ln function is continuous,

2. The ln function is differentiable and $\ln'(x) = \frac{1}{x}$

2.4 SOLVING EQUATIONS INVOLVING ln.

For any positive real numbers a and b :

* $a < b \Leftrightarrow \ln a < \ln b$

* $\ln a < 0 \Leftrightarrow 0 < a < 1$

* $\ln a > 0 \Leftrightarrow a > 1$

We are now able to sketch the table of sign of $\ln x$ **:**

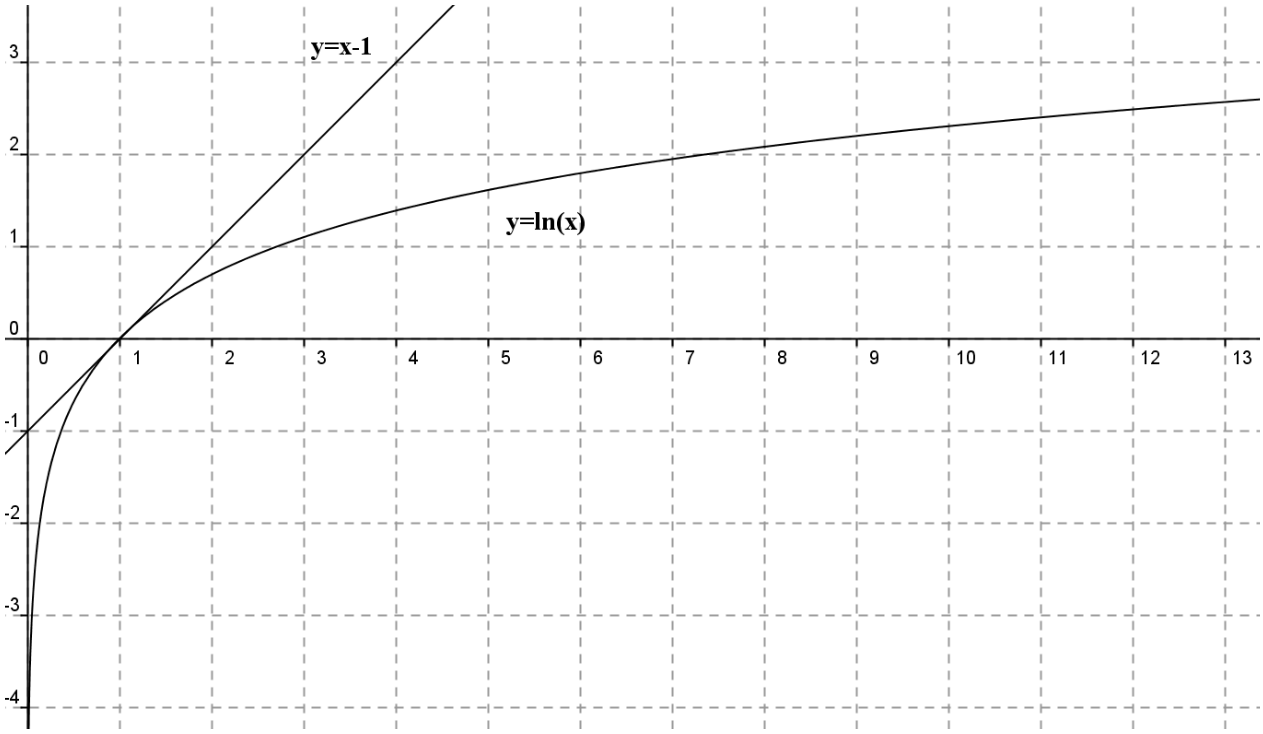
<u>**Exemple :**</u>

Solve a) $\ln^x = -2$ $\qquad$ b) $e^x = 3.5$ $\qquad$ c) $1.05^n = 10$ $\qquad$ d) $x^5 = 5.5$

3. GRAPH.

Table of variation and graph

x	0	1	$+\infty$
$\ln'(x)$	$\|\|$	$+$	
$\ln(x)$	$\|\|$ $-\infty$	$\nearrow$	$+\infty$



Exercises :

1. ***Find the tangent at (1;0) and prove the graph is always below***
2. ***Find the tangent at (e;1) and prove it passes through the origin***

Key points of the chapter

- ✓ Knowing the derivative, variations and the graph of the logarithm function
- ✓ Use the characteristic property and its consequences
- ✓ Solving equations in the form $x^n = k$