PROBABILITIES II
CONTINUOUS PROBABILITY DISTRIBUTIONS
(FR: LOIS DE PROBABILITE CONTINUES)

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à Manosque
Tle ES

1. Definitions

Position of the problem :

Up to now, we have only considered probability distributions on finite sample spaces. What if it's not the case anymore ?

Eg : You choose randomly a number between 0.2 and 0.6. What is the probability that it is in-between 0.3 and 0.35 ?

Definition (probability distribution on a bounded interval) :
Let $I=[a;b]$ ***be a bounded interval and f a continuous and positive function on I.***
A probability distribution P on I has for density f if :

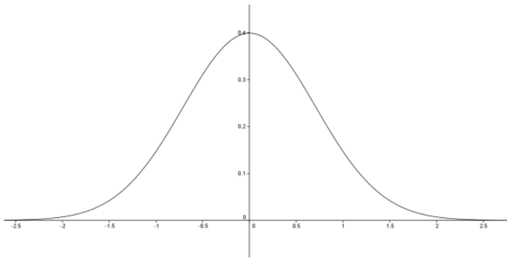
$$P(I)=\int_a^b f(x)\,dx=1$$

$$P([c;d])=\int_c^d f(x)\,dx$$

Notes :

1. **f is often denoted p.d.f (probability density function) in English (fr: fonction de densité)**
2. **Graphical Interpretation :**

$P([c;d])$ represents the area below the graph of the density function between c and d and $P(I)$ (area below the graph of the density function between a and b) is taken as the unit area.



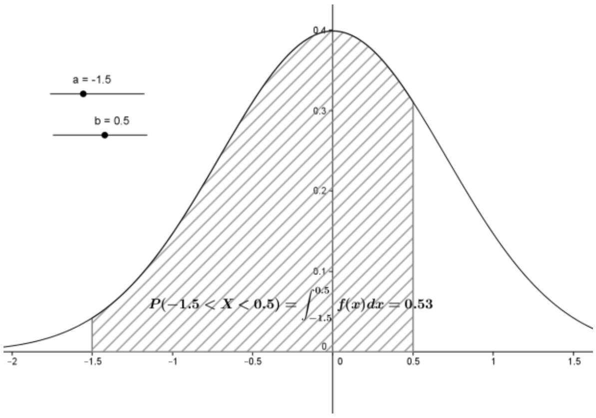
Note : make the link between $P(I)=\int_a^b f(x)\,dx=1$ and $\sum P_i x_i=1$ for discrete probability distributions

3. **If a random variable X follows a distribution P, we can write** $P(c\le X\le d)=\int_c^d f(x)\,dx$

This definition can be widened to a non-bounded interval :

Definition (probability distribution on an interval in the form $I=[a;+\infty[$ ***) :***
Let I be an interval in the form $I=[a;+\infty[$ ***and f*** ***<u>continuous and positive</u> function on I.***
A probability distribution P on I has for <u>density</u> f if :

- $P(I)=\lim\limits_{t\to+\infty}\left(\int_a^t f(x)\,dx\right)=1$
- $P([c;d])=\int_c^d f(x)\,dx$
- $P([c;+\infty[)=1-\int_a^c f(x)\,dx$



Note :

- $P(\{\alpha\})=0$ for any α in I. (since $P(\{\alpha\})=\int_{\alpha}^{\alpha} f(x)dx=0$)
- we hence have $P([c;d])=P([c;d[)=P(]c;d])=P(]c;d[)$

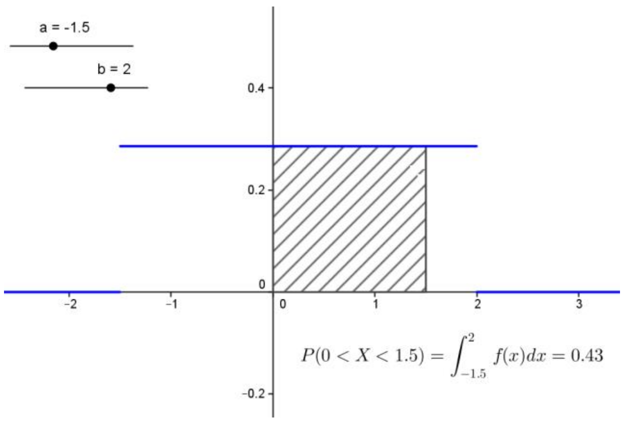
> ***Properties :***
>
> ***- The expected value of a continuous random variable X is :*** $E(X)=\int_a^b xf(x)dx$
>
> ***- If K and J are two intervals contained in I such that*** $P(J)\neq 0$***. We can define the conditional probability*** $P_J(K)=\dfrac{P(J\cap K)}{P(J)}$ ***of K given J.***
>
> ***- If K and J are a partition of I, then*** $P(J\cup K)=P(J)+P(K)=1$***.***

Notes :

1. Be careful using $P(J\cup K)=P(J)+P(K)-P(J\cap K)$, $P(J\cap K)$ can be zero (if $J\cap K$ is finite) without J and K being disjoint.
2. We sometime use the cumulative distribution function (CDF) (fr: function de répartition) defined as :
   $F(x)=P(X\leq x)=\int_{-\infty}^{x} f(x)dx$

2. The uniform (or rectangular) distribution on [a ;b]

> ***Definition :***
>
> ***The uniform distribution on*** $[a;b]$ ***, denoted*** $\mathcal{U}_{[a;b]}$ ***is defined by having for density function f the function equal to :*** $\begin{cases} \dfrac{1}{b-a} & \text{if } a\leq x\leq b \\ 0 & \text{elsewhere} \end{cases}$***.***



Note : - It is the one involved when choosing randomly a number between a and b.

- For any c and d in $[a;b]$, $P(X\in[c;d])=\int_c^d \dfrac{1}{b-a}dx=\dfrac{d-c}{b-a}$

Examples :

a) Calculate $P_{[0.2;0.6]}([0.5;0.55])$

b) You choose randomly a number between 0 and 1 . Given it is between 0.6 and 0.7 , what is the probability for it to be bigger than 0.68 ?

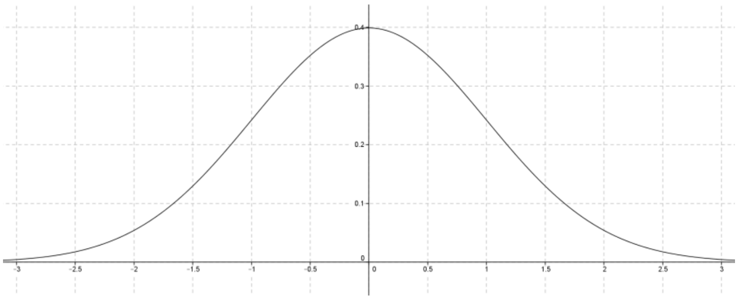
> ***Property:***
>
> ***The expected value of a continuous random variable*** $X\sim\mathcal{U}_{[a;b]}$ ***is :*** $\dfrac{a+b}{2}$

Proof : $E(X)=\int_a^b \dfrac{x}{b-a}dx=....$

3. The standard normal (or Gaussian) distribution (fr: loi normale centrée réduite)

Definition :

A random variable X follows a standard normal distribution when its density function is defined on $\mathbb{R}$ by :

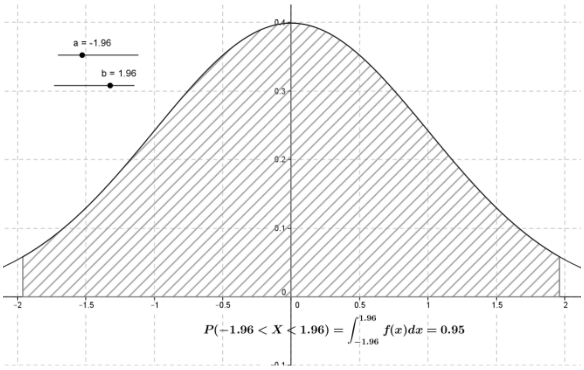


$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

We write $X \sim \mathcal{N}(0;1)$

Notes :

1. This graph is known as the **Gaussian curve** or the bell-shaped graph.



Properties:

If $X \sim \mathcal{N}(0;1)$ ***then :***

$P(1{,}96 \leq X \leq 1{,}96) \approx 0.95$

The expected value of X is $\mu = 0$

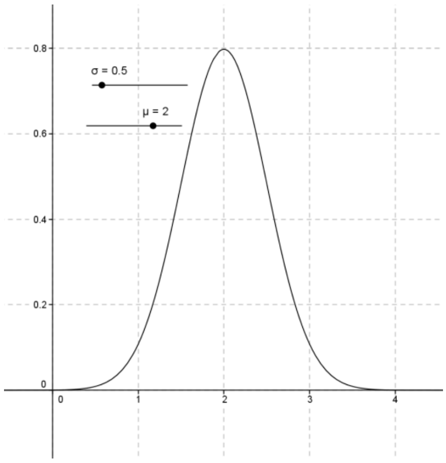
The standard deviation of X is $\sigma = 1$

4. The normal (or Gaussian) distribution (fr: loi normale)

Definition :

A random variable X follows a normal distribution with expected value μ and standard deviation σ when the random variable $\frac{X-\mu}{\sigma}$ follows a standard normal distribution.

We write $X \sim \mathcal{N}(\mu;\sigma^2)$



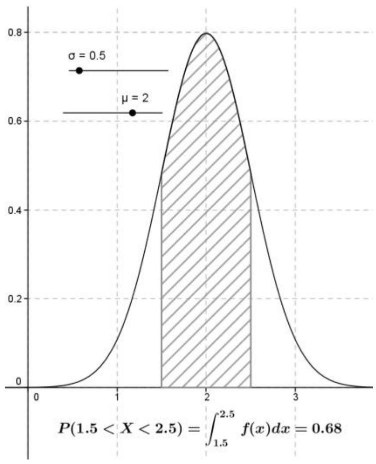
Notes :

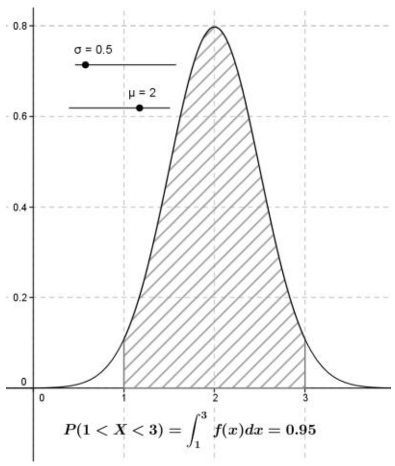
1. The density function is then : $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$
2. μ picks out the peak of the graph and σ its spread (small values lead to tall and narrow graphs, larger values give short, fat graphs).

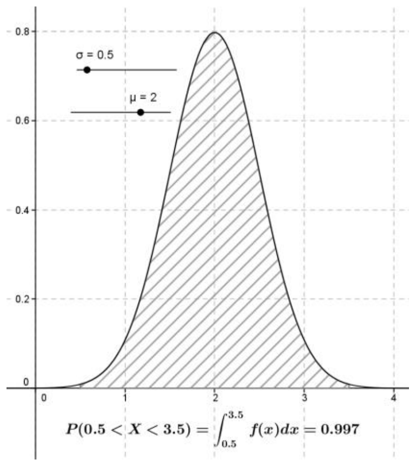
Properties:

If $X \sim \mathcal{N}\left(\mu;\sigma^2\right)$ ***then :***

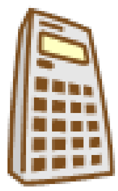
$P(\mu-\sigma \leq X \leq \mu+\sigma) \approx 0.68$ $\quad P(\mu-2\sigma \leq X \leq \mu+2\sigma) \approx 0.95$ $\quad P(\mu-3\sigma \leq X \leq \mu+3\sigma) \approx 0.99$







Note :



1. Since we can't calculate directly (we don't know a primitive) , the calculator is very helpful to calculate approximate values of $P(a \leq X \leq b)$ with the function **normalFrép** .

   Ex : if $X \sim \mathcal{N}(100;25)$, then $P(70 \leq X \leq 95) =$ **normalFrép**(70,95,100,5)

2. With a spreadsheet : $P(X \leq a) =$ LOI.NORMALE.N(a; μ ; σ ;VRAI) and LOI.NORMALE.INVERSE.N(k; μ ; σ) gives the number a such that $P(X \leq a) = k$.

Bullet points of the chapter

- ✓ Dealing with the uniform and normal probability distributions
- ✓ Knowing the graphs of their density functions
- ✓ Knowing that If $X \sim \mathcal{N}(0;1)$ then $P(1{,}96 \leq X \leq 1{,}96) \approx 0.95$
- ✓ Knowing that If $X \sim \mathcal{N}\left(\mu;\sigma^2\right)$ then $P(\mu-\sigma \leq X \leq \mu+\sigma) \approx 0.68$
- ✓ Knowing that If $X \sim \mathcal{N}\left(\mu;\sigma^2\right)$ then $P(\mu-2\sigma \leq X \leq \mu+2\sigma) \approx 0.95$
- ✓ Knowing that If $X \sim \mathcal{N}\left(\mu;\sigma^2\right)$ then $P(\mu-3\sigma \leq X \leq \mu+3\sigma) \approx 0.99$
- ✓ Using a calculator or a spreadsheet to get $P(a \leq X \leq b)$