OXFORD UNIVERSITY

MATHEMATICS, JOINT SCHOOLS AND COMPUTER SCIENCE

WEDNESDAY 3 NOVEMBER 2010

Time allowed: $2\frac{1}{2}$ hours

For candidates applying for Mathematics, Mathematics & Statistics, Computer Science, Mathematics & Computer Science, or Mathematics & Philosophy

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Write your name, test centre (where you are sitting the test), Oxford college (to which you have applied or been assigned) and your proposed course (from the list above) in BLOCK CAPITALS.

NOTE: Separate sets of instructions for both candidates and test supervisors are provided, which should be read carefully before beginning the test.

NAME:

TEST CENTRE:

OXFORD COLLEGE (if known):

DEGREE COURSE:

DATE OF BIRTH:

FOR TEST SUPERVISORS USE ONLY:

[ ] **Tick here if special arrangements were made for the test.**

Please either include details of special provisions made for the test and the reasons for these in the space below or securely attach to the test script a letter with the details.

Signature of Invigilator ______________________

FOR OFFICE USE ONLY:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Total

1. For **ALL APPLICANTS**.

For each part of the question on pages 3–7 you will be given four possible answers, just one of which is correct. Indicate for each part **A**–**J** which answer (a), (b), (c), or (d) you think is correct with a tick (✓) in the corresponding column in the table below. *Please show any rough working in the space provided between the parts.*

	(a)	(b)	(c)	(d)
A				
B				
C				
D				
E				
F				
G				
H				
I				
J				

A. The values of k for which the line $y = kx$ intersects the parabola $y = (x-1)^2$ are precisely

(a) $k \leqslant 0$, (b) $k \geqslant -4$, (c) $k \geqslant 0$ or $k \leqslant -4$, (d) $-4 \leqslant k \leqslant 0$.

B. The sum of the first $2n$ terms of

$$1, 1, 2, \frac{1}{2}, 4, \frac{1}{4}, 8, \frac{1}{8}, 16, \frac{1}{16}, \ldots$$

is

(a) $2^n + 1 - 2^{1-n}$, (b) $2^n + 2^{-n}$, (c) $2^{2n} - 2^{3-2n}$, (d) $\dfrac{2^n - 2^{-n}}{3}$.

Turn Over

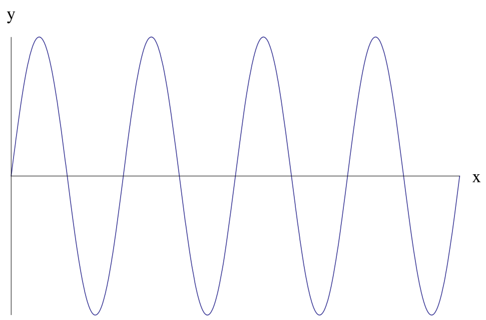
C. In the range $0 \leqslant x < 2\pi$, the equation

$$\sin^2 x + 3\sin x \cos x + 2\cos^2 x = 0$$

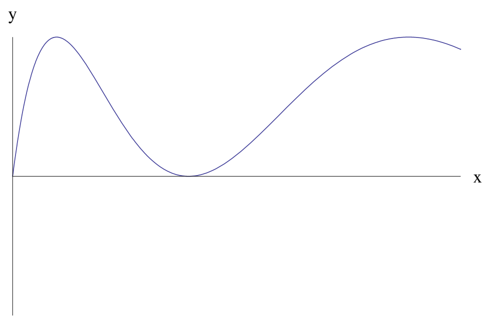
has

(a) 1 solution, (b) 2 solutions, (c) 3 solutions, (d) 4 solutions.

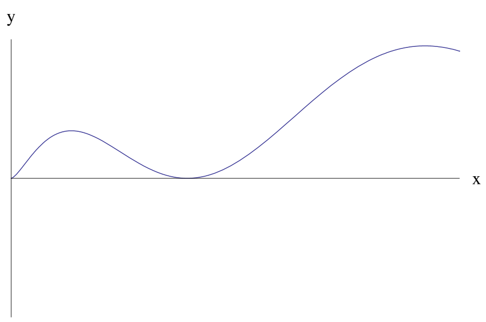
D. The graph of $y = \sin^2 \sqrt{x}$ is drawn in



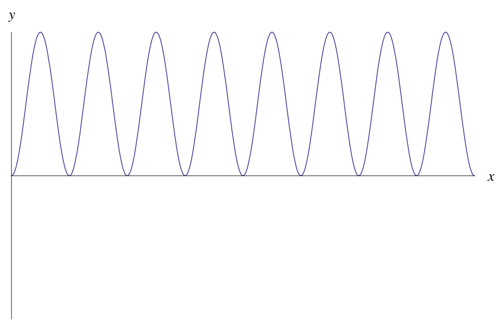
(a)



(b)



(c)

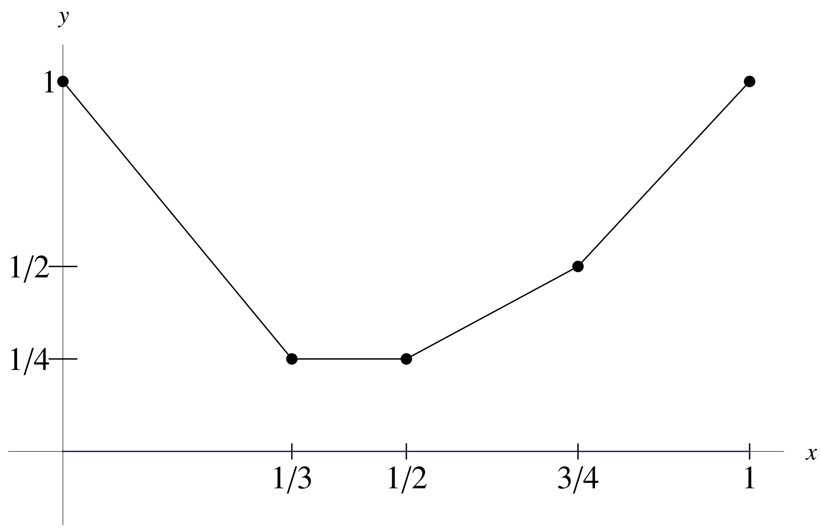


(d)

E. Which is the largest of the following four numbers?

(a) $\log_2 3$, (b) $\log_4 8$, (c) $\log_3 2$, (d) $\log_5 10$.

F. The graph $y = f(x)$ of a function is drawn below for $0 \leqslant x \leqslant 1$.



The trapezium rule is then used to estimate

$$\int_0^1 f(x)\,\mathrm{d}x$$

by dividing $0 \leqslant x \leqslant 1$ into n equal intervals. The estimate calculated will equal the actual integral when

(a) n is a multiple of 4;
(b) n is a multiple of 6;
(c) n is a multiple of 8;
(d) n is a multiple of 12.

Turn Over

G. The function f, defined for whole positive numbers, satisfies $f(1) = 1$ and also the rules

$$\begin{aligned} f(2n) &= 2f(n), \\ f(2n+1) &= 4f(n), \end{aligned}$$

for all values of n. How many numbers n satisfy $f(n) = 16$?

(a) 3, (b) 4, (c) 5, (d) 6.

H. Given a positive integer n and a real number k, consider the following equation in x,

$$(x-1)(x-2)(x-3) \times \cdots \times (x-n) = k.$$

Which of the following statements about this equation is true?

(a) If $n = 3$, then the equation has no real solution x for some values of k.
(b) If n is even, then the equation has a real solution x for any given value of k.
(c) If $k \geqslant 0$ then the equation has (at least) one real solution x.
(d) The equation never has a repeated solution x for any given values of k and n.

I. For a positive number a, let

$$I(a) = \int_0^a \left(4 - 2^{x^2}\right) \, \mathrm{d}x.$$

Then $\mathrm{d}I/\mathrm{d}a = 0$ when a equals

(a) $\dfrac{1+\sqrt{5}}{2}$, (b) $\sqrt{2}$, (c) $\dfrac{\sqrt{5}-1}{2}$, (d) 1.

J. Let a, b, c be positive numbers. There are *finitely* many *positive whole* numbers x, y which satisfy the inequality

$$a^x > c\, b^y$$

if

(a) $a > 1$ or $b < 1$.
(b) $a < 1$ or $b < 1$.
(c) $a < 1$ and $b < 1$.
(d) $a < 1$ and $b > 1$.

Turn Over

2. For **ALL APPLICANTS.**

Suppose that a, b, c are *integers* such that

$$a\sqrt{2} + b = c\sqrt{3}.$$

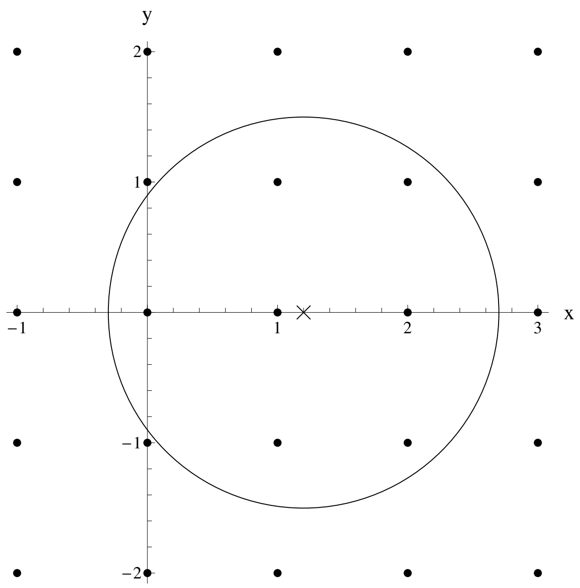
(i) By squaring both sides of the equation, show that $a = b = c = 0$.

[You may assume that $\sqrt{2}, \sqrt{3}$ and $\sqrt{2/3}$ are all irrational numbers. An irrational number is one which cannot be written in the form p/q where p and q are integers.]

(ii) Suppose now that m, n, M, N are *integers* such that the distance from the point (m, n) to $(\sqrt{2}, \sqrt{3})$ equals the distance from (M, N) to $(\sqrt{2}, \sqrt{3})$.

Show that $m = M$ and $n = N$.

Given real numbers a, b and a positive number r, let $N(a, b, r)$ be the number of integer pairs x, y such that the distance between the points (x, y) and (a, b) is less than or equal to r. For example, we see that $N(1.2, 0, 1.5) = 7$ in the diagram below.



(iii) Explain why $N(0.5, 0.5, r)$ is a multiple of 4 for any value of r.

(iv) Let k be any positive integer. Explain why there is a positive number r such that

$$N\left(\sqrt{2}, \sqrt{3}, r\right) = k.$$

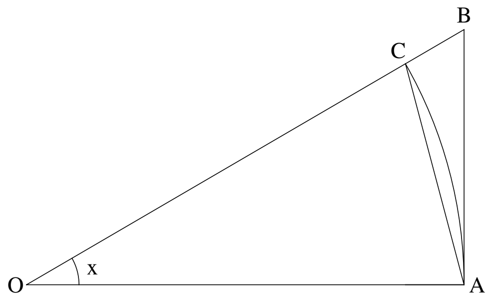
Turn Over

3.

For **APPLICANTS IN** $\left\{ \begin{array}{l} \textbf{MATHEMATICS} \\ \textbf{MATHEMATICS \& STATISTICS} \\ \textbf{MATHEMATICS \& PHILOSOPHY} \\ \textbf{MATHEMATICS \& COMPUTER SCIENCE} \end{array} \right\}$ **ONLY.**

Computer Science **applicants should turn to page 14.**

[In this question, you may assume that the derivative of $\sin x$ is $\cos x$.]



(i) In the diagram above OA and OC are of length 1 and subtend an angle x at O. The angle BAO is a right angle and the circular arc from A to C, centred at O, is also drawn.

By consideration of various areas in the above diagram, show, for $0 < x < \pi/2$, that

$$x \cos x < \sin x < x.$$

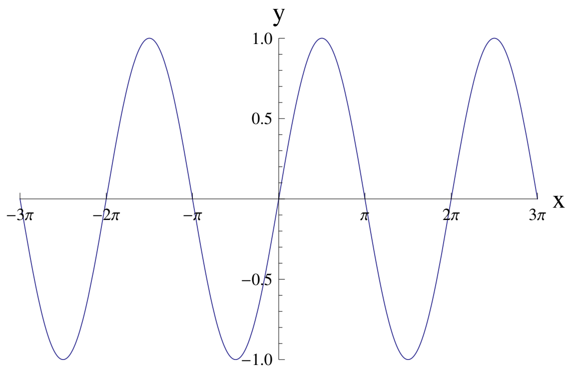
(ii) Sketch, on the axes provided on the opposite page, the graph of

$$y = \frac{\sin x}{x}, \qquad 0 < x < 4\pi.$$

Justify your value that y takes as x becomes small.

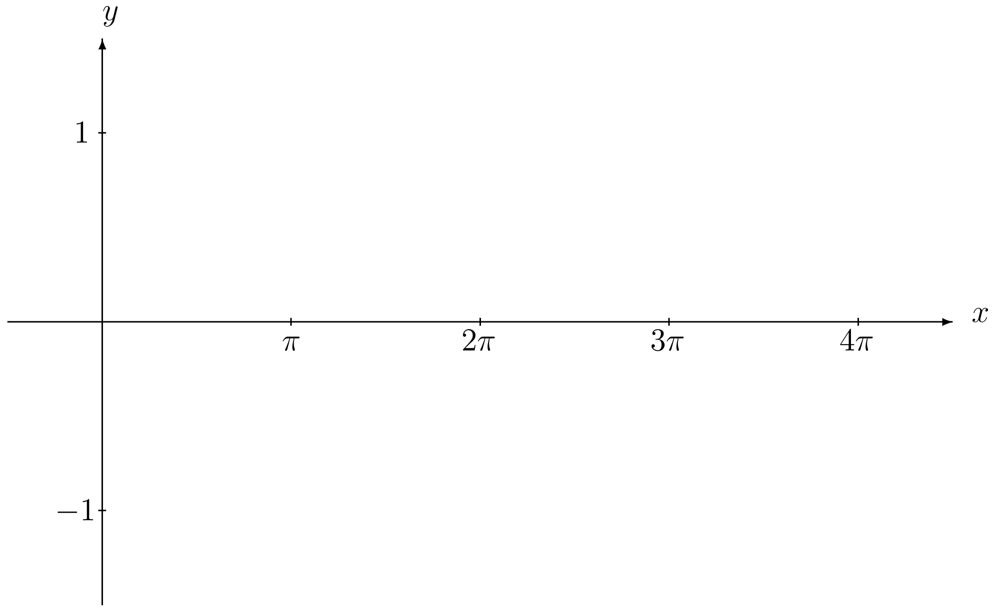
[You do not need to determine the coordinates of the turning points.]

(iii) Drawn below is a graph of $y = \sin x$. Sketch on the same axes the line $y = cx$ where $c > 0$ is such that the equation $\sin x = cx$ has *exactly 5 solutions.*



(iv) Draw the line $y = c$ on the axes on the opposite page.

(v) If X is the largest of the five solutions of the equation $\sin x = cx$, explain why $\tan X = X$.



Turn Over

4.

For **APPLICANTS IN** $\left\{\begin{array}{l}\textbf{MATHEMATICS}\\ \textbf{MATHEMATICS \& STATISTICS}\\ \textbf{MATHEMATICS \& PHILOSOPHY}\end{array}\right\}$ **ONLY.**

Mathematics & Computer Science **and** *Computer Science* **applicants should turn to page 14.**

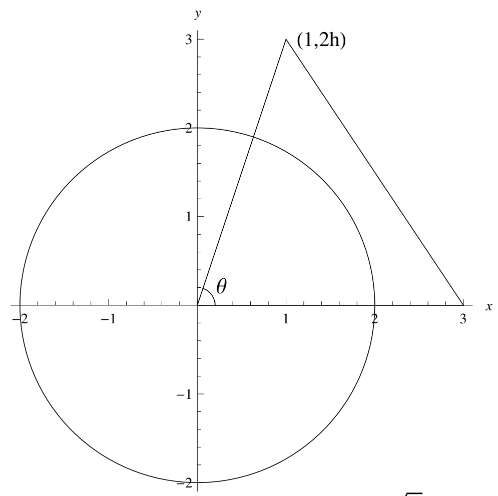


Diagram when $h > 2/\sqrt{5}$

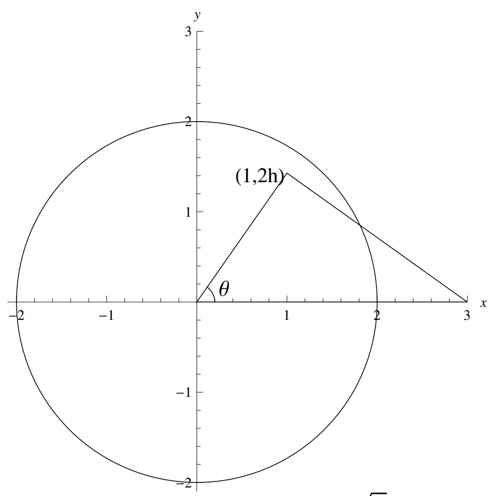


Diagram when $h < \sqrt{3}/2$

The three corners of a triangle T are $(0,0)$, $(3,0)$, $(1,2h)$ where $h > 0$. The circle C has equation $x^2 + y^2 = 4$. The angle of the triangle at the origin is denoted as θ. The circle and triangle are drawn in the diagrams above for different values of h.

(i) Express $\tan\theta$ in terms of h.

(ii) Show that the point $(1,2h)$ lies inside C when $h < \sqrt{3}/2$.

(iii) Find the equation of the line connecting $(3,0)$ and $(1,2h)$. Show that this line is tangential to the circle C when $h = 2/\sqrt{5}$.

(iv) Suppose now that $h > 2/\sqrt{5}$. Find the area of the region inside both C and T in terms of θ.

(v) Now let $h = 6/7$. Show that the point $(8/5, 6/5)$ lies on both the line (from part (iii)) and the circle C.

Hence show that the area of the region inside both C and T equals

$$\frac{27}{35} + 2\alpha$$

where α is an angle whose tangent, $\tan\alpha$, you should determine.

[You may use the fact that the area of a triangle with corners $(0,0)$, (a,b), (c,d) equals $\frac{1}{2}\left|ad - bc\right|$.]

Turn Over

5. For **ALL APPLICANTS.**

This question concerns calendar dates of the form

$$d_1d_2/m_1m_2/y_1y_2y_3y_4$$

in the order day/month/year.

The question specifically concerns those dates which contain no repetitions of a digit. For example, the date 23/05/1967 is one such date but 07/12/1974 is not such a date as both $1 = m_1 = y_1$ and $7 = d_2 = y_3$ are repeated digits.

We will use the Gregorian Calendar throughout (this is the calendar system that is standard throughout most of the world; see below.)

(i) Show that there is no date with no repetition of digits in the years from 2000 to 2099.

(ii) What was the last date before today with no repetition of digits? Explain your answer.

(iii) When will the next such date be? Explain your answer.

(iv) How many such dates were there in years from 1900 to 1999? Explain your answer.

[The Gregorian Calendar uses 12 months, which have, respectively, 31, 28 or 29, 31, 30, 31, 30, 31, 31, 30, 31, 30 and 31 days. The second month (February) has 28 days in years that are not divisible by 4, or that are divisible by 100 but not 400 (such as 1900); it has 29 days in the other years (leap years).]

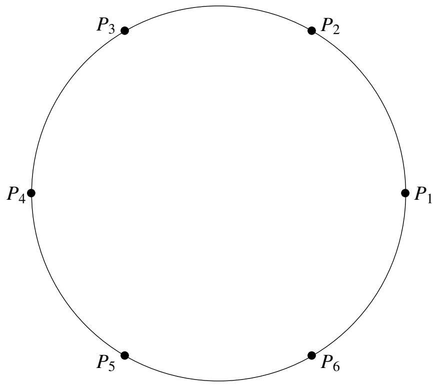
Turn Over

6.

For **APPLICANTS IN** $\left\{\begin{array}{l}\textbf{COMPUTER SCIENCE} \\ \textbf{MATHEMATICS \& COMPUTER SCIENCE}\end{array}\right\}$ **ONLY.**

In the questions below, the people involved make statements about each other. Each person is either a *saint* (S) who always tells the truth or a *liar* (L) who always lies.

(i) Six people, $P_1, P_2, \ldots, P_6$ sit in order around a circular table with P_1 sitting to P_6's right, as shown in the diagram below.



(a) Suppose all six people say "the person directly opposite me is telling the truth". One possibility is that all six are lying. But, in total, how many different possibilities are there? Explain your reasoning.

(b) Suppose now that all six people say "the person to my left is lying". In how many different ways can this happen? Explain your reasoining.

(ii) Now n people $Q_1, Q_2, \ldots, Q_n$ sit in order around a circular table with Q_1 sitting to Q_n's right.

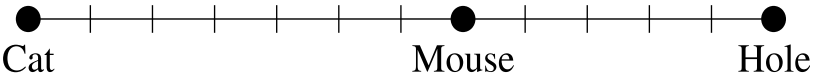
(a) Suppose that all n people make the statement "the person on my left is lying *and* the person on my right is telling the truth". Explain why everyone is lying.

(b) Suppose now that every person makes the statement "either the people to my left and right are both lying *or* both are telling the truth". If at least one person is lying, show that n is a multiple of three.

Turn Over

7. For **APPLICANTS IN COMPUTER SCIENCE ONLY.**

In a game of *Cat and Mouse*, a cat starts at position 0, a mouse starts at position m and the mouse's hole is at position h. Here m and h are integers with $0 < m < h$. By way of example, a starting position is shown below where $m = 7$ and $h = 12$.



With each turn of the game, one of the mouse or cat (but not both) advances one position towards the hole *on the condition that the cat is always strictly behind the mouse and never catches it.* The game ends when the mouse reaches the safety of its hole at position h.

This question is about calculating the number, $g(h, m)$, of different sequences of moves that make a game of Cat and Mouse.

Let C denote a move of the cat and M denote a move of the mouse. Then, for example, $g(3, 1) = 2$ as MM and MCM are the only possible games. Also $CMCCM$ is *not* a valid game when $h = 4$ and $m = 2$ as the mouse would be caught on the fourth turn.

(i) Write down the five valid games when $h = 4$ and $m = 2$.

(ii) Explain why $g(h, h-1) = h - 1$ for $h \geqslant 2$.

(iii) Explain why $g(h, 2) = g(h, 1)$ for $h \geqslant 3$.

(iv) By considering the possible first moves of a game, explain why

$$g(h, m) = g(h, m+1) + g(h-1, m-1) \qquad \text{when } 1 < m < h - 1.$$

(v) Below is a table with certain values of $g(h, m)$ filled in. Complete the remainder of the table and verify that $g(6, 1) = 42$.

m \ h	2	3	4	5	6
5					5
4				4	
3			3		
2		2			
1	1				

End of Last Question