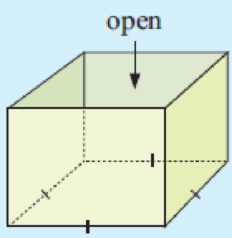
DERIVATIVE'S APPLICATIONS
EXERCICES

<u>3 examples to refresh your memories (try to do them and check the answers on the next pages):</u>

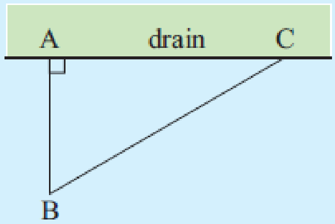
Example 18

Find the most economical shape (minimum surface area) for a box with a square base, vertical sides and an open top, given that it must contain 4 litres.



Example 19

An animal enclosure is a right angled triangle with one leg being a drain. The farmer has 300 m of fencing available for the other two sides, AB and BC.



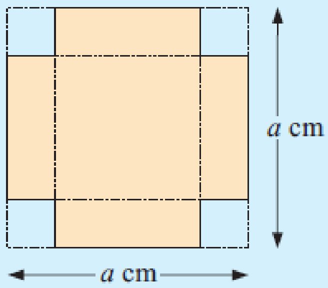
a Show that $AC = \sqrt{90\,000 - 600x}$ if $AB = x$ m.

b Find the maximum area of the triangular enclosure.
Hint: If the area is A m^2, find A^2 in terms of x.
A is a maximum when A^2 takes its maximum value.

Example 20

A square sheet of metal has smaller squares cut from its corners as shown.

What sized square should be cut out so that when the sheet is bent into an open box it will hold the maximum amount of liquid?



Further exercises :

1 A manufacturer can produce x fittings per day where $0 \leqslant x \leqslant 10\,000$. The costs are:

- €1000 per day for the workers
- €2 per day per fitting
- €$\frac{5000}{x}$ per day for running costs and maintenance.

How many fittings should be produced daily to minimise costs?

2 For the cost function $C(x) = 720 + 4x + 0.02x^2$ dollars and price function $p(x) = 15 - 0.002x$ dollars, find the production level that will maximise profits.

3 The total cost of producing x blankets per day is $\frac{1}{4}x^2 + 8x + 20$ dollars, and for this production level each blanket may be sold for $(23 - \frac{1}{2}x)$ dollars.
How many blankets should be produced per day to maximise the total profit?

4 The cost of running a boat is £$\frac{v^2}{10}$ per hour where v is the speed of the boat.
All other costs amount to £62.50 per hour. Find the speed which will minimise the total cost per kilometre.

5 A duck farmer wishes to build a rectangular enclosure of area 100 m^2. The farmer must purchase wire netting for three of the sides as the fourth side is an existing fence. Naturally, the farmer wishes to minimise the length (and therefore cost) of fencing required to complete the job.

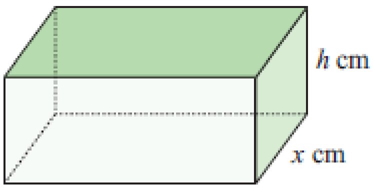
a If the shorter sides have length x m, show that the required length of wire netting to be purchased is $L = 2x + \frac{100}{x}$.

b Use *technology* to help you sketch the graph of $y = 2x + \frac{100}{x}$.

c Find the minimum value of L and the corresponding value of x when this occurs.

d Sketch the optimum situation showing all dimensions.

6 Radioactive waste is to be disposed of in fully enclosed lead boxes of inner volume 200 cm^3. The base of the box has dimensions in the ratio 2 : 1.



a What is the inner length of the box?

b Explain why $x^2h = 100$.

c Explain why the inner surface area of the box is given by $A(x) = 4x^2 + \frac{600}{x}$ cm^2.

d Use *technology* to help sketch the graph of $y = 4x^2 + \frac{600}{x}$.

e Find the minimum inner surface area of the box and the corresponding value of x.

f Sketch the optimum box shape showing all dimensions.

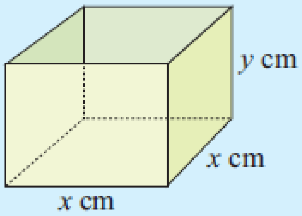
Corrections

Remember :

1. The writing $\dfrac{df(x)}{dx}$ stands for $f'(x)$.
2. The "second derivative test" (consisting in checking that the second derivative is not zero) is equivalent to check the first derivative changes its sign while cancelling to 0 and thus that the function has an extrema.

Example 18

Step 1:



Let the base lengths be x cm and the depth be y cm. The volume

$$V = \text{length} \times \text{width} \times \text{depth}$$

$$\therefore \quad V = x^2 y$$

$$\therefore \quad 4000 = x^2 y \;\; .... \;\; (1) \quad \{\text{as } 1 \text{ litre} \equiv 1000 \text{ cm}^3\}$$

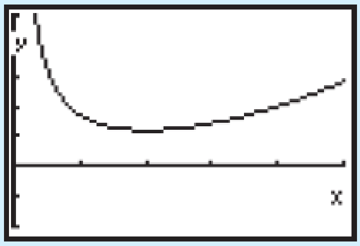
Step 2: The total surface area

$$A = \text{area of base} + 4\,(\text{area of one side})$$

$$= x^2 + 4xy$$

$$= x^2 + 4x\left(\frac{4000}{x^2}\right) \qquad \{\text{using (1)}\}$$

$$\therefore \quad A(x) = x^2 + 16\,000x^{-1} \quad \text{where } x > 0$$



Step 3:

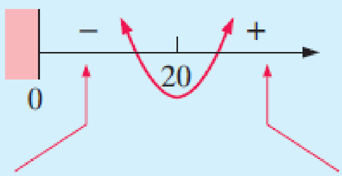
$$A'(x) = 2x - 16\,000x^{-2}$$

$$\therefore \quad A'(x) = 0 \quad \text{when} \quad 2x = \frac{16\,000}{x^2}$$

$$\therefore \quad 2x^3 = 16\,000$$

$$\therefore \quad x = \sqrt[3]{8000} = 20$$

Step 4: **Sign diagram test** *or* **Second derivative test**



if $x = 10$

$$A'(10) = 20 - \tfrac{16\,000}{100} = 20 - 160 = -140$$

if $x = 30$

$$A'(30) = 60 - \tfrac{16\,000}{900} \approx 60 - 17.8 \approx 42.2$$

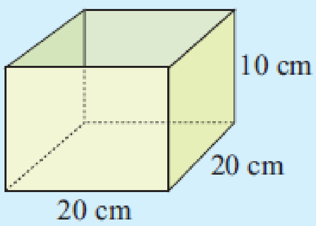
Second derivative test:

$$A''(x) = 2 + 32\,000x^{-3} = 2 + \frac{32\,000}{x^3}$$

which is always positive as $x^3 > 0$ for all $x > 0$.

Both tests establish that the minimum material is used to make the container when $x = 20$ and $y = \dfrac{4000}{20^2} = 10$.

So,



is the most economical shape.

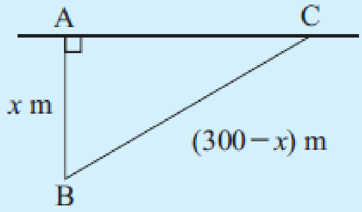
Example 19

a $(AC)^2 + x^2 = (300 - x)^2$ {Pythagoras}

$\therefore \quad (AC)^2 = 90\,000 - 600x + x^2 - x^2$

$= 90\,000 - 600x$

$\therefore \quad AC = \sqrt{90\,000 - 600x}$



b The area of triangle ABC is

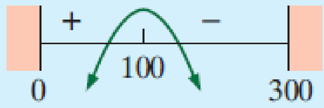
$$A(x) = \tfrac{1}{2}(\text{base} \times \text{altitude})$$
$$= \tfrac{1}{2}(AC \times x)$$
$$= \tfrac{1}{2}x\sqrt{90\,000 - 600x}$$

$0 < x < 300$

$$\therefore \quad [A(x)]^2 = \frac{x^2}{4}(90\,000 - 600x) = 22\,500x^2 - 150x^3$$

$$\therefore \quad \frac{d}{dx}[A(x)]^2 = 45\,000x - 450x^2$$
$$= 450x(100 - x)$$

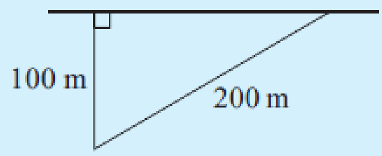
with sign diagram:



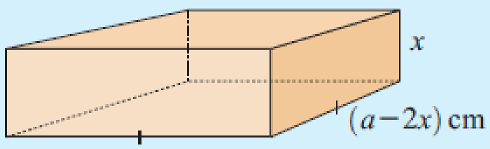
$A(x)$ is maximised when $x = 100$

so $A_{max} = \tfrac{1}{2}(100)\sqrt{90\,000 - 60\,000}$

$\approx 8660 \text{ m}^2$



Example 20



Let x cm by x cm squares be cut out.

Volume = length × width × depth

$= (a - 2x) \times (a - 2x) \times x$

$\therefore \quad V(x) = x(a - 2x)^2$

Now $V'(x) = 1(a - 2x)^2 + x \times 2(a - 2x)^1 \times (-2)$ {product rule}

$= (a - 2x)[a - 2x - 4x]$

$= (a - 2x)(a - 6x)$

$\therefore \quad V'(x) = 0$ when $x = \dfrac{a}{2}$ or $\dfrac{a}{6}$

However, $a - 2x$ must be > 0 and so $x < \dfrac{a}{2}$

Thus $x = \dfrac{a}{6}$ is the only value in $0 < x < \dfrac{a}{2}$ with $V'(x) = 0$.

Second derivative test:

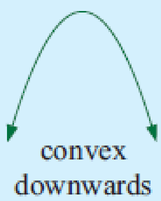
Now $V''(x) = -2(a - 6x) + (a - 2x)(-6)$ {product rule}

$= -2a + 12x - 6a + 12x$

$= 24x - 8a$

$\therefore \quad V''\left(\dfrac{a}{6}\right) = 4a - 8a = -4a$ which is < 0

$\therefore$ the volume is maximised when $x = \dfrac{a}{6}$.



Conclusion: When $x = \dfrac{a}{6}$, the resulting container has maximum capacity.