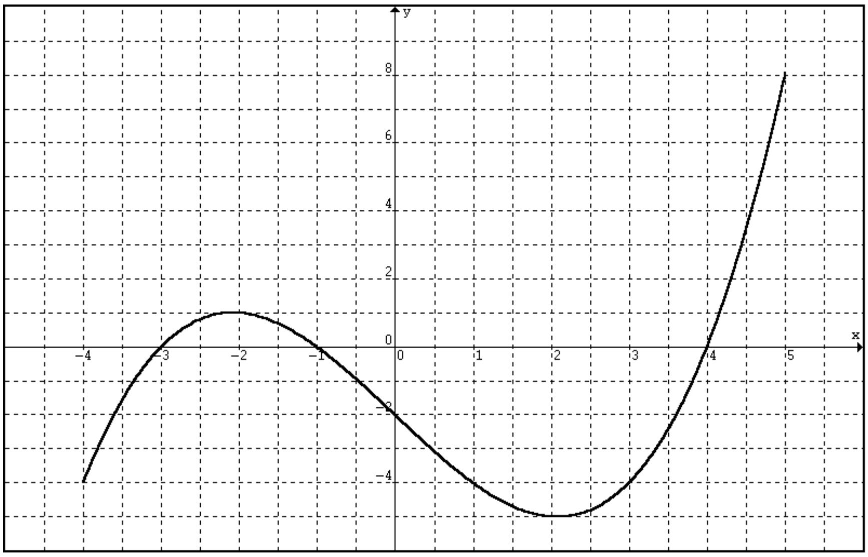
DIFFERENTIATION
(FR: DERIVATION)

1. NOTION OF TANGENT TO A GRAPH



1. The graph opposite represents the function f.

Draw what you think is the tangent to the curve for the point with x co-ordinate 3.

Compare your drawing with your neighbour's.

Can you try to define « a tangent to a graph at a point with x co-ordinate a »

2. Draw the tangent to the graph where the x co-ordinate is 0. Compare with your neighbour. Can you find the gradient with certitude?

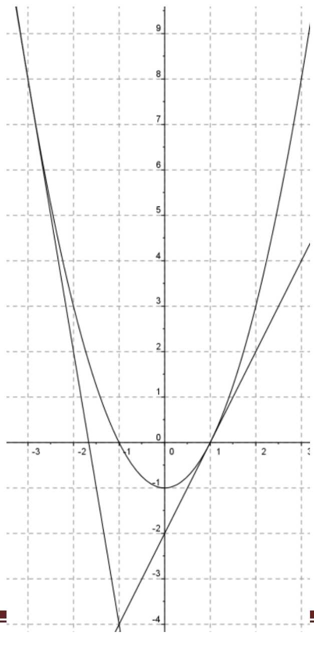
3. Draw the tangent to the curve at a point with x co-ordinate 2. Compare with your neighbour. Can you find the gradient ?

2. GRAPHICAL APPROACH TO THE NOTION OF DIFFERENTIATION:

<u>Definitions:</u> f is a function defined on an interval I. a is an element of I, and $\mathcal{C}$ the graph of f on I.
We say that the function f is differentiable at a (Fr: dérivable en a) if and only if the graph of f has a non-vertical tangent at the point (a, f(a)).
In this case, the gradient (Fr: coefficient directeur) of the tangent is called the derivative (Fr: nombre dérivé) at a and is written f '(a).

Examples:

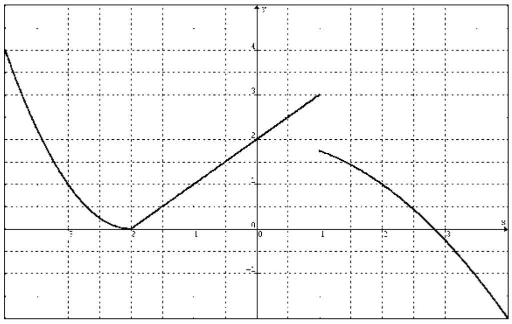
f is the function defined on ℝ by: $f(x) = x^2 - 1$



f is differentiable at -3 and f'(-3)=-6

f is differentiable at 1 and f'(1)= 2

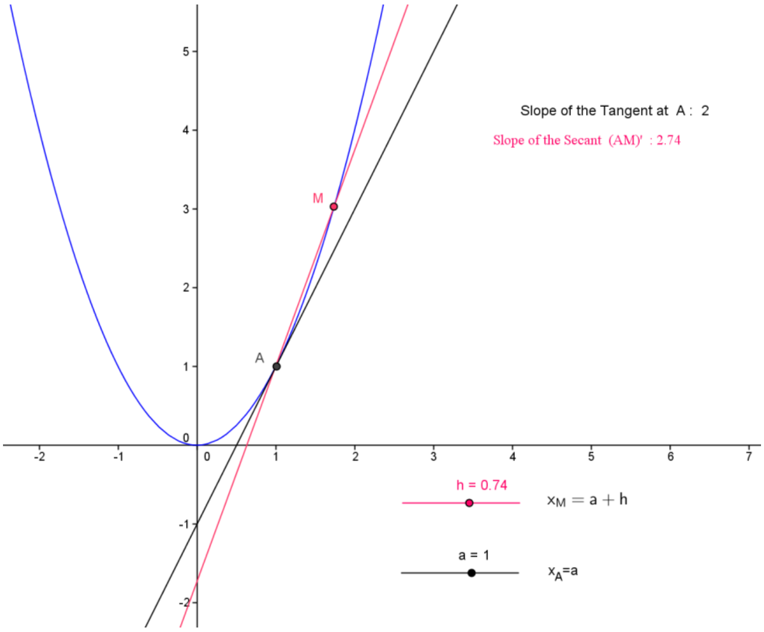
- g is defined on [-5, 8] by the graph:



g is not differentiable at -2 nor at 1.

3. APPROACH TO DIFFERENTIATION BY THE NOTION OF LIMIT:

> ***Definition 1:*** ***f is a function defined on an interval I and a an element of I.***
> ***The average increase of f (Fr: accroissement moyen) between a and a+h is:***
> $$\frac{f(a+h)-f(a)}{h}$$



Note:
The average rate of increase between a and a+h can be thought of as the gradient of the line passing through the points (a, f(a)) and (a+h, f(a+h)).

> ***Definition 2 :*** ***The derivative of f at a is :***
> $$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=f'(a)$$

Example :
Let's prove $f'(1)=2$ for the function $f(x)=x^2$

4. DIFFERENTIABLE FUNCTIONS:

> ***Definition :*** ***f is a function defined on an interval I. f is said to be differentiable (Fr: dérivable) on I if f is differentiable for all x in I.***

note : Differentiability of reference functions:
The reference functions (except the absolute value function) are differentiable on all intervals of their domain.

Note :
The domains in the first table are the biggest domains on which the function is differentiable.

TABLE OF DERIVATIVES		
On	f(x)	f'(x)
$\mathbb{R}$	$x \mapsto a$	$x \mapsto 0$
$\mathbb{R}$	$x \mapsto x$	$x \mapsto 1$
$\mathbb{R}$	$x \mapsto ax+b$	$x \mapsto a$
$\mathbb{R}$	$x \mapsto x^n$	$x \mapsto nx^{n-1}$
$\mathbb{R}^*$	$x \mapsto \frac{1}{x}$	$x \mapsto -\frac{1}{x^2}$
$\mathbb{R}^*$	$x \mapsto \frac{1}{x^n}$	$x \mapsto -\frac{n}{x^{n+1}}$
$\mathbb{R}^{+*}$	$x \mapsto \sqrt{x}$	$x \mapsto \frac{1}{2\sqrt{x}}$

4.1 ALGEBRA OF DIFFERENTIATION

If two functions f and g are differentiable on an interval I then :

- The sum and the product of these two functions are differentiable on I;
- The product of the function f and a real k is a function differentiable on I;
- If g never equals zero on I, the quotient f/g is differentiable on I;

ALGEBRA
$(kf)' = k.f'$
$(f+g)' = f' + g'$
$(fg)' = f'g + fg'$
$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$
$\left(\frac{1}{g}\right)' = \frac{-g'}{g^2}$

Examples :

4.2 VARIATIONS OF A FUNCTION :

<u>Fundamental Theorem:</u> For f a function differentiable on an interval I
If f is increasing on I then f '(x) is positive on I
If f is decreasing on I then f '(x) is négative on I
If f is constant on I, then f '(x) is zero for all x in I.

<u>Converse Theorem :</u> For a function f defined on an interval I
If, for all x in I, f'(x) > 0 then f is strictly increasing on I.
If, for all x in I, f'(x) < 0 then f is strictly decreasing on I.
If, for all x in I, f'(x) = 0 then f is constant on I.

Consequence

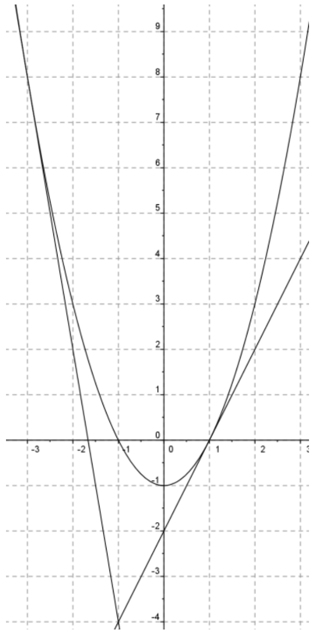
<u>Theorem :</u> If f is a differentiable function on an interval I and if for a value a of I, "the derivative f' of f cancels and changes its signs at a", then f has a turning point with x co-ordinate a.

example :

Method for studying the variations of a function :

eg $f(x) = 2x^2 - 3x + 4$

5. TANGENT TO A CURVE :



If f is a function differentiable on an interval I and **a** a point of I, then the derivative f'(**a**) represents the slope of the tangent at the point with co-ordinates (**a**, f(**a**)). As we know this point belongs to the tangent, we are able to find the equation of this tangent :

> ***Property :***
> ***The equation of the tangent at a is:*** $y = f'(a).(x-a)+f(a)$.

Proof :

Example :

6. LINKS WITH ECONOMICS:

For a function of time f	
Absolute increase of f between a and b	$f(b)-f(a)$
Average increase of f between a and b	$\dfrac{f(b)-f(a)}{b-a}$
Instantaneous increase of f at a	$f'(a)$
Relative increase of f between a and b	$\dfrac{f(b)-f(a)}{f(a)}$

For a function of total cost of a quandity q : $q \mapsto C(q)$	
Average cost	$C_M(q) = \dfrac{C(q)}{q}$
Marginal cost	$C_m(q) \approx C'(q)$

Let's consider the size of the population of a given country which can be represented with respect to time by a function f : $f(t)$ is thus the size of the population (let's say in millions) at time t. Let's assume that $f(t) = 0{,}1t^2 - 1{,}5t + 9$.

- ✓ The **variation (or absolute variation)** of f between the two different times t_1 and t_2 is $\mathbf{f}(t_2)-\mathbf{f}(t_1)$. Its value is in millions (like f).
  Ex: in our case the increase of f between 0 and 10 is $\mathbf{f}(10)-\mathbf{f}(0) = (10-15+9)-9 = -5$ millions

- ✓ The **average variation** of f between the two different times t_1 and t_2 is $\dfrac{\mathbf{f}(t_2)-\mathbf{f}(t_1)}{t_2-t_1}$.
  Its value is in millions per year (you divide millions of people by years)

Ex: in our case the average increase of f between 0 and 10 is $\frac{\mathbf{f}(10)-\mathbf{f}(0)}{10-0}=\frac{-5}{10}=-0,5$ millions/year meaning that the country has lost, on average, 0.5 million people a year between year 0 and 10.

- ✓ The **relative variation** (or rate of change) of f between the two different times t_1 and t_2 is $\frac{\mathbf{f}(t_2)-\mathbf{f}(t_1)}{\mathbf{f}(t_1)}$.

Its value has no units (since you divide millions of people by millions of people). It is usually put in %.

Ex: in our case the rate of change of f between 0 and 10 is $\frac{\mathbf{f}(10)-\mathbf{f}(0)}{\mathbf{f}(0)}=\frac{-5}{9}\approx-0,56\approx-56\%$ meaning that the country has lost 56% of its population between year 0 and 10.

- ✓ **The instantaneous variation** of f at a particular time t_1 is $\mathbf{f}'(t_1)$. It is the limit of the average variation of f between the times t_1 and t_2 as t_2 approaches t_1.Its value is in millions per year (like the average variation).

Ex: in our case the rate of change of f at year 10 is $\mathbf{f}'(10)=0,2\times10-1,5=18,5$ meaning that the population of the country rises at the speed of 18.5 million a year.