Probabilities I

"Le hasard, c'est Dieu qui se promène incognito" A.EINSTEIN

1. VOCABULARY

Random Experiment (Fr : Expérience aléatoire) : A set of precise conditions describing an experiment whose outcome cannot be predicted with certitude.

(Universal) sample space (Fr : Univers) : The set of all the possible results or outcomes (Fr :**éventualités**) in a random experiment. It is often written *S*, Ω, or *U* (Fr: **Ω** or E).

Note : It often happens that several sample spaces are possible for the same experiment. It is best to choose that for which all the outcomes are **equiprobable** (Fr: **équiprobables) ie equally likely.**

Event (Fr : evènement) : A set of outcomes, (so a subset of the sample space). If the event contains only one possible outcome, it is called an **elementary event** (Fr: **d'événement élémentaire)**.

If the event is **Ω** itself, it is called an **exhaustive event** (Fr: **événement certain)**.

If the event is the empty set, it is called an **impossible event (Fr: événement impossible)**.

Example:

Random experiment : Picking randomly one card from a full (52) pack of playing cards (13 of each of diamonds, clubs, hearts and spades).

The sample space is the 52 cards set. Assuming all the outcomes are equally likely, we can calculate probabilities of such events :

Event A is the event of selecting an ace: p(A) = 4/52 = 1/13 (as all the cards are

Event B is the event of selecting a black card : p(B)=26/52 = 1/2

The event of selecting a diamond king is an elementary event.

The event of selecting a card is an exhaustive event.

The event of selecting a card which is both red and black is an impossible event.

Event "A and B" : This corresponds to **"A∩B"**. It is the intersection of subsets A and B, ie the elements that both subsets have in common. In other words, "A and B" occurs if and only if both A and B occurs.

In our example, A∩B is the event of selecting a black ace. p (A∩B)=2/52 = 1/26

Event "A or B" : This corresponds to **"A∪B".** It is the union of the two subsets, ie the event which groups all the events contained in A with all those contained in B. In other words, "A or B" occurs if and only if A occurs or B occurs or both A and B occur.

In our example, A∪B is the event of selecting a black card or a (red) ace. p (A∪B)=(26+2)/52 = 28/52=7/13

Event [not A] or A' (Fr: $\overline{A}$): This is the **complement** (Fr: **l'évènement contraire)** of A. It corresponds to all elements in **Ω** which are not in A. In other words, A' occurs if A does not.

In our example, $\overline{A}$ is the event of selecting a card that is not black (ie a red one) : p ($\overline{A}$)= 26/52 = ½

$\overline{B}$ is the event of selecting a card that is not an ace : p ($\overline{B}$)= 48/52 = 12/13

A Venn diagram shows a summary of the sample space.

Mutually exclusive events (Fr: Evènements incompatibles) : Two events A and B are said to be **mutually exclusive** if they correspond to two disjoint subsets of Ω. In other words, they cannot both occur at the same time. $A \cap B = \varnothing$

Drawing a picture card and drawing a 2 are mutually exclusive.

Special case : A and $\overline{A}$, complementary events, are mutually exclusive.

Attention! Mutually exclusive events are not necessarily complementary.

2. CALCULATING PROBABILITIES:

Ω is a sample space associated with a random experiment.
For all subset A of Ω, the real number p(A) is called the **"probability of event A".**

Probability Laws (Fr: propriétés) :

- The probability of an event is a real number between 0 and 1.
- The probability of an event is equal to the sum of the probabilities of the elementary events it is made up of.
- The sum of the probabilities of all the elementary events is 1.
- The probability of an exhaustive event is 1. $p(\Omega) = 1$
- The probability of an impossible event is 0. $p(\varnothing) = 0$
- If A and B are **two mutually exclusive events** then : $p(A \cup B) = p(A) + p(B)$. *(Addition law)*
- **In the general case we have:** $p(A \cup B) = p(A) + p(B) - p(A \cap B)$
- If A and A' are complementary events then : $p(A')=1 - p(A)$

Equiprobability (Fr : Equiprobabilité) :

In the case where all the elementary events have the same probability, we say that they are **equally likely** (Fr: **équiprobables**).

When calculating probabilities, it is often useful to use a **tree diagram** , which shows a summary of all possible events.

Example : A player throws, 3 times in a row, a unfair coin (Head appears 2 times out of 5). If a gets at least one Head, he wins, unless he loses. What is probability he wins ?

3. RANDOM VARIABLES (FR : VARIABLES ALEATOIRES)

Definitions :

1. We say that we define a **random variable** X (over the sample space Ω of a random experiment), when we associate one number to each outcome of Ω.
2. Defining the **probability distribution** (or law) (fr: loi de probabilité) of X consists in associating each value x_i of X with a positive number p_i such that : $\sum_{i=1}^{n} p_i = 1$.

Note:

1. In fact $p_i = P(X = x_i)$
2. A probability distribution is usually given in the form of a table with the different values of x_i and the corresponding p_i.
3. Quite often the random variable is associated with the gain playing a game.

Values of X	x_1	x_2	.....	x_n
Probability	p_1	p_2	.....	p_n

Example: We toss a coin and throw a die.

If you get Tail or 1 or 2 you win 1€. If you get Head and 5 or 6 you lose 3€. Otherwise, you don't gain nor lose anything. Let's denote X the gain. What is the probability distribution of X ?
X can take 3 values : 0,1 and -3.
The sample space of the experiment has 12 equally likely outcomes (like (H,1),(T,5)…).
The event (X=-3) is composed of the outcomes (H,5) and (H,6). The event (X=0) is composed of the outcomes (H,4) and (H,3). The probability distribution of X is thus :

Values of X	0	1	-3
Probability	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{6}$

Definition : **Expected value** (fr: Espérance)
The expected value of a random variable X , denoted E(X) is the mean value of the x_i , weighted by their probabilities p_i. $E(X) = \sum_{i=1}^{n} x_i p_i = x_1 p_1 + x_2 p_2 + ... + x_n p_n$

Note:

It represents the mean value of X you can expect if you perform the random experiment a large number of times.

Example: In the previous game, $E(X) = 0 \times \frac{1}{6} + 1 \times \frac{2}{3} - 3 \times \frac{1}{6} = \frac{1}{6}$. If you play this game a large number of time, you will gain $\frac{1}{6}$€ by game, on average.