BACCALAUREAT GENERAL

Jeudi 20 décembre 2012

MATHEMATIQUES **- série S –**

Durée de l'épreuve : 4 heures ***Coefficient : 7***

Les calculatrices électroniques de poche sont autorisées.

Le candidat doit traiter les quatre exercices

Exercice 1. (5 pts) from Baccalauréat S Antilles-Guyane juin 2012

Let's consider the function f defined on $\mathbb{R}$ by $f(x) = xe^{x-1} + 1$ and $\mathcal{C}$ its graph in an orthonormal coordinate system $(O, \vec{i}, \vec{j})$ with unit 2cm.

Part A : Study of the function

1. Find the limit of f at $-\infty$. What can you then deduce for $\mathcal{C}$?
2. Find the limit of f at $+\infty$.
3. We take as given that f is differentiable on $\mathbb{R}$. Calculate its derivative and prove that $f'(x) = (x+1)e^{x-1}$
4. Study the variations of f on $\mathbb{R}$ and draw up its table of variations.
5. a. Prove that there exists a unique real number α such that $f(x) = 3$.
   b. Give an approximate value of α, rounding to the nearest hundredth.

Part B : Study of a special tangent

Let **a** be a strictly positive real number. The aim of this part is to search for a tangent to $\mathcal{C}$ which passes through the point with x-coordinate **a** and also goes through the origin of the coordinate system.

1. We denote T_a the tangent to $\mathcal{C}$ passing through the point with x-coordinate **a**. Work out an equation of T_a.
2. Prove that T_a will pass through the origin if, and only if, **a** is a solution of $1 - x^2e^{x-1} = 0$.
3. *In this question, any trace of your research, even incomplete, will be taken into account in the grade.*
   Prove that 1 is the unique solution of $1 - x^2e^{x-1} = 0$ on $]0;+\infty[$.
4. Thus give an equation of the required tangent.
5. Carefully plot $\mathcal{C}$ and the required tangent. Mark α on your graph.

Exercise 2. (5 pts) Baccalauréat S Nouvelle Calédonie mars 2012

Consider two urns and one fair die with faces are labelled from 1 to 6.
Urn U_1 contains 3 red balls and one black one.
Urn U_2 contains 3 red balls and 2 black ones.
A game takes place the following way : the player rolls the die; if the outcome is 1, he randomly picks a ball from U_1, otherwise he randomly picks a ball from U_2.
We consider the following events :
A: “rolling 1 with the die”
B: “picking a Black ball”

1. a. Draw the tree diagram corresponding to this random experiment.
   b. Prove that $P(B)=\frac{3}{8}$.
   c. Knowing that we have picked a black ball, work out the probability that we have rolled a 1 with the die.

2. We decide a game is won when a black ball is picked. A player plays 10 independent games in a row, replacing the picked ball in its urn after each game. Let X be the random variable equal to the number of games that are won.
   a. Calculate the probability of winning exactly 3 games. Round your result to 3 dp.
   b. Calculate the probability of winning at least one game. Round your result to 3 dp.
   c. You are given the following table :

k	1	2	3	4	5	6	7	8	9	10
$P(X<k)$	0.0091	0.0637	0.2100	0.4467	0.6943	0.8725	0.9616	0.9922	0.9990	0.9999

Let N be an integer between 1 and 10. Consider the event :
“The player wins at least N games”.
From which value of N onwards is $P(N)$ smaller than 0.1 ?

Exercice 3. (5 pts) Baccalauréat S Polynésie juin 2012

Part A

Let's consider the opposite algorithm :
What is the output when $= 3$?

```
INPUT
    Enter the natural (non-zero) number N
PROCESSING
    U takes the value 0
    For k ranging from 0 to N − 1
        U takes the value 3U − 2k + 3
    End For
OUTPUT
    Print U
```

Part B

Let's consider the sequence (u_n) defined by $u_0 = 0$, and for any natural number n : $u_{n+1} = 3u_n - 2n + 3$.

1. Calculate u_1 and u_2.
2. a. Prove by induction that for any natural number n, $u_n \geq n$.

   b. Thus deduce the limit of (u_n).
3. Prove that (u_n) is increasing.
4. Let (v_n) be the sequence defined, for any natural number n, by $v_n = u_n - n + 1$.
   a. Prove that (v_n) is a geometric sequence.
   b. Deduce that $u_n = 3^n + n - 1$.
5. Let p be a non-zero natural number.
   a. Why can we say that there exists at least an integer n_0 such that, for all $n \geq n_0$, $u_n \geq 10^p$?

   We are now interested in the smallest integer n_0.

   b. Justify that $n_0 \leq 3p$.

   c. Using your calculator, find the value of n_0 for $p = 3$.

   d. Suggest an algorithm which, for a given value of p as input, ends up displaying the value n_0 such that, for all $n \geq n_0$, $u_n \geq 10^p$.

Exercice4. (5 pts)

PartA: Study of an auxiliary function

The function g is defined on $\mathbb{R}$ by : $g(x) = 2e^x - 2x + 1$

1. Work out its limit at $-\infty$.

2. Find $\lim\limits_{x \to +\infty} \frac{x}{e^x}$ and deduce the limit of g at $+\infty$.

3. Study the variations of g on $\mathbb{R}$ and draw up its table of variations.

4. Deduce from this the sign of g on $\mathbb{R}$.

Part B: Study of a function f

The function f is defined on $\mathbb{R}$ by : $f(x) = (2x+1)(1+e^{-x})$

We denote $\mathcal{C}$ its graph in an orthonormal coordinate system $\left(O, \vec{i}, \vec{j}\right)$.

1. Study the sign of fon $\mathbb{R}$.

2. Study the limits of f at $-\infty$ and $+\infty$.

3. Calculate $f'(x)$, where f' is the derivative of f, and check that $f'(x)$ and $g(x)$ have the same sign. Draw up the table of variation of f.

4. a) Prove that the line (D) with equation $y = 2x + 1$ is an oblique asymptote to $\mathcal{C}$ at $+\infty$.
   b) Find the relative position of $\mathcal{C}$ and (D)

CORRECTION

Exercice 1. (5 pts) from Baccalauréat S Antilles-Guyane juin 2012

Let's consider the function f defined on $\mathbb{R}$ by $f(x)=xe^{x-1}+1$ and $\mathcal{C}$ its graph in an orthonormal coordinate system $(O,\vec{i},\vec{j})$ with unit 2cm.

Part A : Study of the function

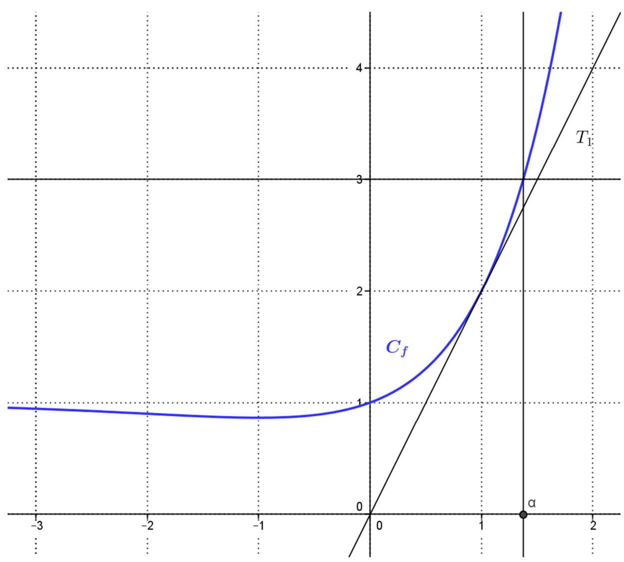
1. As $\lim\limits_{x\to-\infty} xe^{x}=0$ (compared growth) , $\lim\limits_{x\to-\infty} f(x)=1$.
   The line with equation is thus an horizontal asymptote to $\mathcal{C}$
2. As $\lim\limits_{x\to+\infty} x=+\infty$ and $\lim\limits_{x\to+\infty} e^{x}=+\infty$, by product $\lim\limits_{x\to+\infty} f(x)=+\infty$
3. $f'(x)=(u\times v)'=u'v+v'u=e^{x}+xe^{-x}=(1+x)e^{x}$
4. f' has ame sign as 1+x hence the table of variations.

x	$-\infty$		-1		$+\infty$
f'(x)		−	0	+	
f	1 ↘		$1-e^{-2}\approx 0.86$	↗	$+\infty$

5. a. f being continuous on $\mathbb{R}$, we can use the IVT :
   on $]-\infty;-1[$, $f(x)=3$ has no solution since f is strictly decreasing and $f(x)$ ranges from 1 to 0.86
   on $]1;+\infty[$, $f(x)=3$ has one unique solution since f is strictly increasing and $f(x)$ ranges from 0.86 to $+\infty$
   so there exists a unique real number α such that $f(x)=3$.
   b. Give an approximate value of $\alpha\approx 1.37$, rounding to the nearest hundredth.

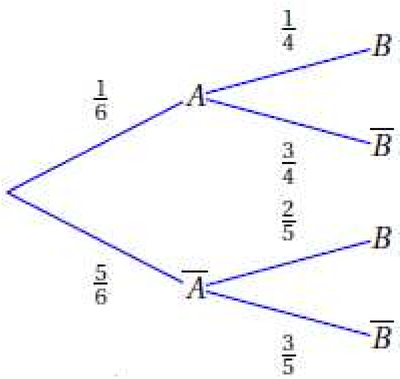
Part B : Study of a special tangent

1. An equation of T_a is $y=f'(a)(x-a)+f(a)=(a+1)e^{a-1}(x-a)+ae^{a-1}+1$
2. T_a will pass through the origin if, and only if, the y-intercept is 0, that's if $-a(a+1)e^{a-1}+ae^{a-1}+1=0$, or $1-a^{2}e^{a-1}=0$.(E)
3. 1 is obviously a solution of $1-x^{2}e^{x-1}=0$.
   Let's denote $g(x)=1-x^{2}e^{x-1}$.
   $g'(x)=...=-x(2+x)e^{x-1}$ is strictly negative on $]0;+\infty[$, so g is strictly decreasing on this interval. As $\lim\limits_{x\to+\infty} g(x)=-\infty<0$ and $g(0)=1>0$, g being continuous, tannks to the IVT there exists a unique number such that $g(x)=0$.
4. Replacing a by 1 in (E), we get : $y=2(x-1)+2=2x$.
5. .



Exercise 2. (5 pts) Baccalauréat S Nouvelle Calédonie mars 2012

1. a.



b. From the “total probability” law

$$P(B)=P(A\cap B)+P\left(\overline{A}\cap B\right)=\frac{1}{6}\times\frac{1}{4}+\frac{5}{6}\times\frac{2}{5}=\frac{3}{8}.$$

c. $P_B(A)=\dfrac{P(A\cap B)}{P(B)}=\dfrac{\frac{1}{6}\times\frac{1}{4}}{\frac{3}{8}}=\ldots=\dfrac{1}{9}.$

2.

a. The games being independent, the random variable X measuring the number of successes got follows a binomial distribution with parameters 10 and $\frac{3}{8}$. Hence

$$P(X=3)=\binom{10}{3}\times\left(\frac{3}{8}\right)^3\times\left(1-\frac{3}{8}\right)^7=120\times\left(\frac{3}{8}\right)^3\times\left(\frac{5}{8}\right)^7=\ldots\approx 0.236$$

b. $P(X\geq 1)=1-P(X=0)=1-\binom{10}{0}\times\left(\frac{3}{8}\right)^0\times\left(\frac{5}{8}\right)^{10}=\ldots\approx 0.991.$

c.

k	1	2	3	4	5	6	7	8	9	10
$P(X<k)$	0.0091	0.0637	0.2100	0.4467	0.6943	0.8725	0.9616	0.9922	0.9990	0.9999

$P(N)=P(X\geq N)=1-P(X<N)$

$P(N)<0.1$ when $P(X<N)>0.9$, so $N=7$

Exercice 3. (5 pts) Baccalauréat S Polynésie juin 2012

Part A

Before entering the for loop : N=3, U=0

At the end of the first loop for k=0 : $U=3\times0-2\times0+3=3$

At the end of the second loop for k=1 : $U=3\times3-2\times1+3=10$

At the end of the third loop for k=2 : $U=3\times10-2\times2+3=29$

```
INPUT
    Enter the natural (non-zero)
number N
PROCESSING
    U takes the value 0
    For k ranging from 0 to N − 1
        U takes the value
3U − 2k + 3
    End For
OUTPUT
    Print U
```

Part B

Let's consider the sequence (u_n) defined by $u_0 = 0$, and for any natural number n : $u_{n+1} = 3u_n - 2n + 3$.

1. $u_1 = 3\times u_0 - 2\times 0 + 3 = 3$ and $u_2 = 3\times u_1 - 2\times 1 + 3 = 10$.

2. a. Let's denote P(n) "$u_n \geq n$" .

   Initialisation : P(0) is true since $u_0 = 0 \geq 0 = n$

   Heredity : Let's assume P(n) is true and prove P(n+1) is true as well :

   P(n) true means $u_n \geq n$, then $u_{n+1} = 3u_n - 2n + 3 \geq 3n - 2n + 3 \geq n + 3 \geq n + 1$ so P(n+1) is true.

   P(n) is then hereditary

   Conclusion : by induction , P(n) is true for any n in $\mathbb{N}$

   b. $u_n \geq n$ and $\lim\limits_{n\to+\infty} n = +\infty$, so by comparison , $\lim\limits_{n\to+\infty} u_n = +\infty$.

3. $u_{n+1} - u_n = 2u_n - 2n + 3 = 2\underbrace{(u_n - n)}_{>0} + 3 > 0$ so (u_n) is increasing.

4. Let (v_n) be the sequence defined, for any natural number n, by $v_n = u_n - n + 1$.

   a. $v_{n+1} = u_{n+1} - (n+1) + 1 = \underbrace{3u_n - 2n + 3}_{u_{n+1}} - n = 3u_n - 3n + 3 = 3(u_n - n + 1) = 3v_n$ so (v_n) is a geometric sequence with common ratio 3.

   b. Know we know that $v_n = v_0 \times q^n = 3^n$ since $v_0 = 1$. Hence $u_n = v_n + n - 1 = 3^n + n - 1$.

5. Let p be a non-zero natural number.

   a. $\lim\limits_{n\to+\infty} u_n = +\infty$, so by definition there exists an integer n_0 such that, for all $n \geq n_0$, u_n exceed any real number A, so specially 10^p .

   b. $u_{3p} = 3^{3p} + 3p - 1 = 27^p + 3p - 1 \geq 27^p \geq 10^p$ so 3p is a value of n such that $u_n \geq 10^p$. The sequence being increasing, if $n \geq 3p$, then $u_n \geq u_{3p} \geq 10^p$. n_0 being the smallest of these values of n, $n_0 \leq 3p$.

   c. $u_6 = 734 < 1000$ and $u_7 = 2193 > 1000$ so $n_0 = 7$ for $p = 3$.

   d.

```
INPUT
    Enter the natural (non-zero) number p
PROCESSING
    U takes the value 0
    k takes the value 0
    While U < 10^p
        U takes the value 3U − 2k + 3
        k takes the value k + 1
    End While
OUTPUT
    Print k
```

Should have reminded you ex 0 on the sequences sheet !

Exercice4. (5 pts)

PartA: Study of an auxiliary function

The function g is defined on $\mathbb{R}$ by : $g(x)=2e^x-2x+1$

1. at $-\infty$:
   $2e^x$ tends to 0 and $-2x+1$ tends to $+\infty$, so by addition $g(x)$ tends to $+\infty$
2. $\lim_{x\to+\infty}\frac{x}{e^x}=0$ since we know that $\lim_{x\to+\infty}\frac{e^x}{x}=+\infty$.
   $g(x)=e^x\left(2-2\underbrace{\frac{x}{e^x}}_{\text{tends to } 0}+\underbrace{\frac{1}{e^x}}_{\text{tends to } 0}\right)$ so the limit of g at $+\infty$ is the same as e^x , so $+\infty$.
3. G is differentiable on $\mathbb{R}$ and $g'(x)=2e^x-2=2\left(e^x-1\right)$. $e^x-1>0$ when $x>0$, hence the table of variation :

x	$-\infty$		0		$+\infty$
g'(x)		$-$	0	$+$	
f	$+\infty$	↘	3	↗	$+\infty$

4. The minimum of g being 3, g is always positive on $\mathbb{R}$.

Part B: Study of a function f

1. $(1+e^{-x})$ being always positive, $f(x)$ as the same sign as $(2x+1)$ that's positive on $[-0.5;+\infty[$ and negative on $]-\infty;-0.5]$
2. Study the limits of f at $-\infty$ and $+\infty$.
   at $-\infty$:
   $2x+1$ tends to $-\infty$ and $(1+e^{-x})$ tends to $+\infty$, so by product $f(x)$ tends to $-\infty$
   at $+\infty$:
   $2x+1$ tends to $+\infty$ like 2x and $(1+e^{-x})$ tends to 0 like e^{-x}, so by product $f(x)$ has the same limit as $2xe^{-x}$ which is 0 (compared growth of x and e^{-x} at $+\infty$).
3. $f'(x)=(u\times v)'=u'v+v'u=2\left(1+e^{-x}\right)-\left(2x+1\right)e^{-x}=\left(1-2x+2e^x\right)e^{-x}=g\left(x\right)e^{-x}$, so $f'(x)$ and $g(x)$ have the same sign (always positive), hence the table of variation of f.

x	$-\infty$		$+\infty$
f'(x)		$+$	
f	$-\infty$	↗	0

4. a) $\varepsilon\left(x\right)=f\left(x\right)-\left(2x+1\right)=\left(2x+1\right)e^{-x}$ whose limit at $+\infty$ is 0, thus that the line (D) with equation $y=\left(2x+1\right)$ is an oblique asymptote to $\mathcal{C}$ at $+\infty$.
   b) $\varepsilon\left(x\right)$ has the sign of $\left(2x+1\right)$, so on $]-\infty;-0.5]$ of $\mathcal{C}$ is below (D) and on $[-0.5;+\infty[$ of $\mathcal{C}$ is above (D).