3 COMPLEX NUMBERS

Objectives

After studying this chapter you should

- understand how quadratic equations lead to complex numbers and how to plot complex numbers on an Argand diagram;
- be able to relate graphs of polynomials to complex numbers;
- be able to do basic arithmetic operations on complex numbers of the form $a+ib$;
- understand the polar form $[r,\theta]$ of a complex number and its algebra;
- understand Euler's relation and the exponential form of a complex number $re^{i\theta}$;
- be able to use de Moivre's theorem;
- be able to interpret relationships of complex numbers as loci in the complex plane.

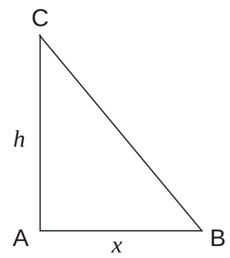
3.0 Introduction

The history of complex numbers goes back to the ancient Greeks who decided (but were perplexed) that no number existed that satisfies

$$\boxed{x^2=-1}$$

For example, *Diophantus* (about 275 AD) attempted to solve what seems a reasonable problem, namely

> 'Find the sides of a right-angled triangle of perimeter 12 units and area 7 squared units.'



Letting $\mathrm{AB}=x$, $\mathrm{AC}=h$ as shown,

then area $=\frac{1}{2}xh$

and perimeter $=x+h+\sqrt{x^2+h^2}$

Activity 1

Show that the two equations above reduce to

$$6x^2 - 43x + 84 = 0$$

when perimeter $= 12$ and area $= 7$. Does this have real solutions?

A similar problem was posed by *Cardan* in 1545. He tried to solve the problem of finding two numbers, a and b, whose sum is 10 and whose product is 40;

i.e. $$a + b = 10 \qquad (1)$$

$$ab = 40 \qquad (2)$$

Eliminating b gives

$$a(10 - a) = 40$$

or $$a^2 - 10a + 40 = 0.$$

Solving this quadratic gives

$$a = \frac{1}{2}(10 \pm \sqrt{-60}) = 5 \pm \sqrt{-15}$$

This shows that there are no **real** solutions, but if it is agreed to continue using the numbers

$$a = 5 + \sqrt{-15},\; b = 5 - \sqrt{-15}$$

then equations (1) and (2) are satisfied.

Show that equations (1) and (2) are satisfied by these values of x and y.

So these are solutions of the original problem but they are not **real** numbers. Surprisingly, it was not until the nineteenth century that such solutions were fully understood.

The square root of -1 is denoted by i, so that

$$i = \sqrt{-1}$$

and $$a = 5 + \sqrt{15}\,i, \quad b = 5 - \sqrt{15}\,i$$

are examples of **complex numbers**.

Activity 2 The need for complex numbers

Solve if possible, the following quadratic equations by factorising or by using the quadratic formula. If a solution is not possible explain why.

(a) $x^2 - 1 = 0$ (b) $x^2 - x - 6 = 0$

(c) $x^2 - 2x - 2 = 0$ (d) $x^2 - 2x + 2 = 0$

You should have found (a), (b) and (c) straightforward to solve but in (d) a term appears in the solution which includes the square root of a **negative** number and to obtain solutions you need to use the symbol $i = \sqrt{-1}$, or

$$\boxed{i^2 = -1}$$

It is then possible to obtain a solution to (d) in Activity 2.

Example

Solve $x^2 - 2x + 2 = 0$.

Solution

Using the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \quad x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)}$$

$$\Rightarrow \quad x = \frac{2 \pm \sqrt{-4}}{2}$$

But $\sqrt{-4} = \sqrt{4(-1)} = \sqrt{4}\sqrt{-1} = 2\sqrt{-1} = 2i$

(using the definition of i).

Therefore $x = \dfrac{2 \pm 2i}{2}$

$$\Rightarrow \quad x = 1 \pm i$$

Therefore the two solutions are

$$x = 1 + i \quad \text{and} \quad x = 1 - i$$

Activity 3

Solve the following equations, leaving your answers in terms of i:

(a) $x^2 + x + 1 = 0$ (b) $3x^2 - 4x + 2 = 0$

(c) $x^2 + 1 = 0$ (d) $2x - 7 = 4x^2$

The set of solutions to a quadratic equation such as

$$ax^2 + bx + c = 0$$

can be related to the intercepts on the x-axis when the graph of the function

$$f(x) = ax^2 + bx + c$$

is drawn.

Activity 4 Quadratic graphs

Using a graphics calculator, a graph drawing program on a computer, a spreadsheet or otherwise, draw the graphs of the following functions and find a connection between the existence or not of real solutions to the related quadratic equations.

(a) $f(x) = x^2 - 1$ (b) $f(x) = x^2 - x - 6$

(c) $f(x) = x^2 - 2x - 2$ (d) $f(x) = x^2 + x + 1$

(e) $f(x) = 3x^2 - 4x + 2$ (f) $f(x) = x^2 + 1$

You should have noted that if the graph of the function either intercepts the x-axis in two places or touches it in one place then the solutions of the related quadratic equation are real, but if the graph does not intercept the x-axis then the solutions are complex.

If the quadratic equation is expressed as $ax^2 + bx + c = 0$, then the expression that determines the type of solution is $b^2 - 4ac$, called the **discriminant**.

In a quadratic equation $ax^2 + bx + c = 0$, if:

$b^2 - 4ac > 0$ then solutions are real and different
$b^2 - 4ac = 0$ then solutions are real and equal
$b^2 - 4ac < 0$ then solutions are complex

3.1 Complex number algebra

A number such as $3+4i$ is called a **complex number**. It is the sum of two terms (each of which may be zero).

The real term (not containing i) is called the **real** part and the coefficient of i is the **imaginary** part. Therefore the real part of $3+4i$ is 3 and the imaginary part is 4.

A number is real when the coefficient of i is zero and is imaginary when the real part is zero.

e.g. $3+0i=3$ is real and $0+4i=4i$ is imaginary.

Having introduced a complex number, the ways in which they can be combined, i.e. addition, multiplication, division etc., need to be defined. This is termed the **algebra** of complex numbers. You will see that, in general, you proceed as in real numbers, but using

$$i^2=-1$$

where appropriate.

But first **equality** of complex numbers must be defined.

If two complex numbers, say

$$a+bi,\ c+di$$

are **equal**, then both their real and imaginary parts are equal;

$$\boxed{a+bi=c+di \quad \Rightarrow \quad a=c \text{ and } b=d}$$

Addition and subtraction

Addition of complex numbers is defined by separately adding real and imaginary parts; so if

$$z=a+bi,\ w=c+di$$

then $$z+w=(a+c)+(b+d)i\,.$$

Similarly for **subtraction**.

Example

Express each of the following in the form $x+yi$.

(a) $(3+5i)+(2-3i)$

(b) $(3+5i)+6$

(c) $7i-(4+5i)$

Solution

(a) $(3+5i)+(2-3i)=3+2+(5-3)i=5+2i$

(b) $(3+5i)+6=9+5i$

(c) $7i-(4+5i)=7i-4-5i=-4+2i$

Multiplication

Multiplication is straightforward provided you remember that $i^2=-1$.

Example

Simplify in the form $x+yi$:

(a) $3(2+4i)$

(b) $(5+3i)i$

(c) $(2-7i)(3+4i)$

Solution

(a) $3(2+4i) = 3(2)+3(4i) = 6+12i$

(b) $(5+3i)\,i = (5)\,i+(3i)\,i = 5i+3\left(i^2\right) = 5i+(-1)3 = -3+5i$

(c) $$\begin{aligned}(2-7i)(3+4i) &= (2)(3) - (7i)(3) + (2)(4i) - (7i)(4i)\\ &= 6-21i+8i-(-28)\\ &= 6-21i+8i+28\\ &= 34-13i\end{aligned}$$

In general, if

$$z = a+bi, \quad w = c+di,$$

then $$\begin{aligned}zw &= (a+bi)(c+di)\\ &= ac - bd + (ad + bc)i\end{aligned}$$

Activity 5

Simplify the following expressions:

(a) $(2+6i)+(9-2i)$ (b) $(8-3i)-(1+5i)$

(c) $3(7-3i)+i(2+2i)$ (d) $(3+5i)(1-4i)$

(e) $(5+12i)(6+7i)$ (f) $(2+i)^2$

(g) i^3 (h) i^4

(i) $(1-i)^3$ (j) $(1+i)^2+(1-i)^2$

(k) $(2+i)^4+(2-i)^4$ (l) $(a+ib)(a-ib)$

Division

The **complex conjugate** of a complex number is obtained by changing the sign of the imaginary part. So if $z=a+bi$, its complex conjugate, $\bar{z}$, is defined by

$$\bar{z}=a-bi$$

Note: an alternative notation often used for the complex conjugate is z^*.

Any complex number $a+bi$ has a complex conjugate $a-bi$ and from Activity 5 it can be seen that $(a+bi)(a-bi)$ is a real number. This fact is used in simplifying expressions where the denominator of a quotient is complex.

Example

Simplify the expressions:

(a) $\dfrac{1}{i}$ (b) $\dfrac{3}{1+i}$ (c) $\dfrac{4+7i}{2+5i}$

Solution

To simplify these expressions you multiply the numerator and denominator of the quotient by the complex conjugate of the denominator.

(a) The complex conjugate of i is $-i$, therefore

$$\frac{1}{i}=\frac{1}{i}\times\frac{-i}{-i}=\frac{(1)(-i)}{(i)(-i)}=\frac{-i}{-(-1)}=-i$$

(b) The complex conjugate of $1+i$ is $1-i$, therefore

$$\frac{3}{1+i}=\frac{3}{1+i}\times\frac{1-i}{1-i}=\frac{3(1-i)}{(1+i)(1-i)}=\frac{3-3i}{2}=\frac{3}{2}-\frac{3}{2}i$$

(c) The complex conjugate of $2+5i$ is $2-5i$ therefore

$$\frac{4+7i}{2+5i} = \frac{4+7i}{2+5i}\times\frac{2-5i}{2-5i} = \frac{43-6i}{29} = \frac{43}{29} - \frac{6}{29}i$$

Activity 6 Division

Simplify to the form $a+ib$

(a) $\frac{4}{i}$ (b) $\frac{1-i}{1+i}$ (c) $\frac{4+5i}{6-5i}$ (d) $\frac{4i}{(1+2i)^2}$

3.2 Solving equations

Just as you can have equations with real numbers, you can have equations with complex numbers, as illustrated in the example below.

Example

Solve each of the following equations for the complex number z.

(a) $4+5i = z-(1-i)$

(b) $(1+2i)z = 2+5i$

Solution

(a) Writing $z = x+iy$,

$$4+5i = (x+yi)-(1-i)$$

$$4+5i = x-1+(y+1)i$$

Comparing real parts $\Rightarrow$ $4 = x-1,\quad x=5$

Comparing imaginary parts $\Rightarrow$ $5 = y+1,\quad y=4$

So $z = 5+4i$. In fact there is no need to introduce the real and imaginary parts of z, since

$$4+5i = z-(1-i)$$

$$\Rightarrow \quad z = 4+5i+(1-i)$$

$$\Rightarrow \quad z = 5+4i$$

(b) $(1+2i)z = 2+5i$

$$z = \frac{2+5i}{1+2i}$$

$$z = \frac{2+5i}{1+2i} \times \frac{1-2i}{1-2i}$$

$$z = \frac{12+i}{5} = \frac{12}{5} + \frac{1}{5}i$$

Activity 7

(a) Solve the following equations for real x and y

(i) $3+5i+x-yi=6-2i$

(ii) $x+yi=(1-i)(2+8i)$.

(b) Determine the complex number z which satisfies $z(3+3i)=2-i$.

Exercise 3A

1. Solve the equations:

(a) $x^2+9=0$ (b) $9x^2+25=0$

(c) $x^2+2x+2=0$ (d) $x^2+x+1=0$

(e) $2x^2+3x+2=0$

2. Find the quadratic equation which has roots $2 \pm \sqrt{3}i$.

3. Write the following complex numbers in the form $x+yi$.

(a) $(3+2i)+(2+4i)$ (b) $(4+3i)-(2+5i)$

(c) $(4+3i)+(4-3i)$ (d) $(2+7i)-(2-7i)$

(e) $(3+2i)(4-3i)$ (f) $(3+2i)^2$

(g) $(1+i)(1-i)(2+i)$

4. Find the value of the real number y such that $(3+2i)(1+iy)$ is (a) real (b) imaginary.

5. Simplify:

(a) i^3 (b) i^4 (c) $\frac{1}{i}$ (d) $\frac{1}{i^2}$ (e) $\frac{1}{i^3}$

6. If $z=1+2i$, find

(a) z^2 (b) $\frac{1}{z}$ (c) $\frac{1}{z^2}$

7. Write in the form $x+yi$:

(a) $\frac{2+3i}{1+i}$ (b) $\frac{-4+3i}{-2-i}$ (c) $\frac{4i}{2-i}$

(d) $\frac{1}{2+3i}$ (e) $\frac{3-2i}{i}$ (f) $\frac{p+qi}{r+si}$

8. Simplify:

(a) $\frac{(2+i)(3-2i)}{1+i}$ (b) $\frac{(1-i)^3}{(2+i)^2}$

(c) $\frac{1}{3+i} - \frac{1}{3-i}$

9. Solve for z when

(a) $z(2+i)=3-2i$ (b) $(z+i)(1-i)=2+3i$

(c) $\frac{1}{z} + \frac{1}{2-i} = \frac{3}{1+i}$

10. Find the values of the real numbers x and y in each of the following:

(a) $\frac{x}{1+i} + \frac{y}{1-2i} = 1$

(b) $\frac{x}{2-i} + \frac{yi}{i+3} = \frac{2}{1+i}$

11. Given that p and q are real and that $1+2i$ is a root of the equation

$$z^2+(p+5i)z+q(2-i)=0$$

determine:

(a) the values of p and q;

(b) the other root of the equation.

12. The complex numbers u, v and w are related by

$$\frac{1}{u}=\frac{1}{v}+\frac{1}{w}.$$

Given that $v=3+4i, w=4-3i$, find u in the form $x+iy$.

3.3 Argand diagram

Any complex number $z=a+bi$ can be represented by an ordered pair (a, b) and hence plotted on xy-axes with the real part measured along the x-axis and the imaginary part along the y-axis. This graphical representation of the complex number field is called an **Argand diagram**, named after the Swiss mathematician *Jean Argand* (1768-1822).

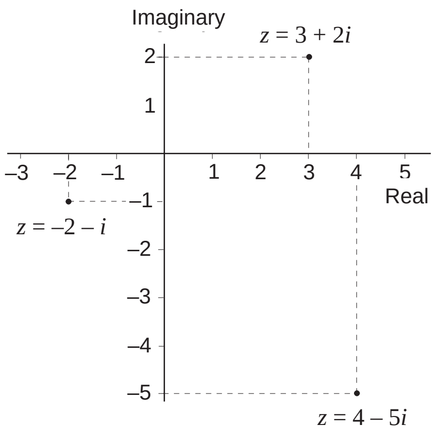
Example

Represent the following complex numbers on an Argand diagram:

(a) $z=3+2i$ (b) $z=4-5i$ (c) $z=-2-i$

Solution

The Argand diagram is shown opposite.



Activity 8

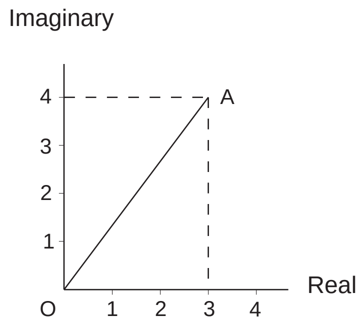
Let $z_1=5+2i$, $z_2=1+3i$, $z_3=2-3i$, $z_4=-4-7i$.

(a) Plot the complex numbers z_1, z_2, z_3, z_4 on an Argand diagram and label them.

(b) Plot the complex numbers z_1+z_2 and z_1-z_2 on the same Argand diagram. Geometrically, how do the positions of the numbers z_1+z_2 and z_1-z_2 relate to z_1 and z_2?

3.4 Polar coordinates

Consider the complex number $z = 3 + 4i$ as represented on an Argand diagram. The position of A can be expressed as coordinates (3, 4), the cartesian form, or in terms of the length and direction of OA.



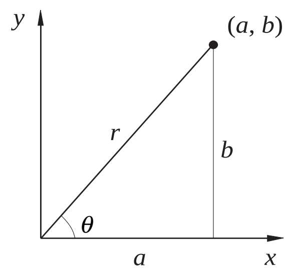
Using Pythagoras' theorem, the length of OA $= \sqrt{3^2 + 4^2} = 5$.

This is written as $|z| = r = 5$. $|z|$ is read as the **modulus** or absolute value of *z*.

The angle that OA makes with the positive real axis is

$$\theta = \tan^{-1}\left(\frac{4}{3}\right) = 53.13° \text{ (or } 0.927 \text{ radians)}.$$

This is written as $\arg(z) = 53.13°$. You say $\arg(z)$ is the **argument** or phase of *z*.



The parameters $|z|$ and $\arg(z)$ are in fact the equivalent of **polar coordinates** r, θ as shown opposite. There is a simple connection between the polar coordinate form and the cartesian or rectangular form (*a*, *b*):

$$a = r\cos\theta, \quad b = r\sin\theta.$$

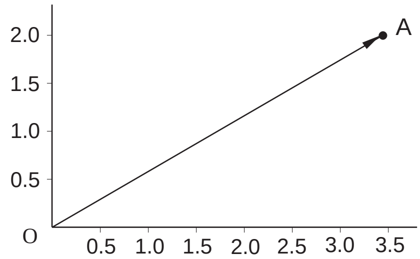
Therefore

$$z = a + bi = r\cos\theta + r\,i\sin\theta = r(\cos\theta + i\sin\theta)$$

where $|z| = r$, and $\arg(z) = \theta$.

It is more usual to express the angle θ in radians. Note also that it is convention to write the *i* before $\sin\theta$, i.e. $i\sin\theta$ is preferable to $\sin\theta\, i$.

In the diagram opposite, the point A could be labelled $(2\sqrt{3},\ 2)$ or as $2\sqrt{3} + 2i$.



The angle that OA makes with the positive *x*-axis is given by

$$\theta = \tan^{-1}\left(\frac{2}{2\sqrt{3}}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right).$$

Therefore $\theta = \dfrac{\pi}{6}$ or $2\pi + \dfrac{\pi}{6}$ or $4\pi + \dfrac{\pi}{6}$ or ... etc. There is an infinite number of possible angles. The one you should normally use is in the interval $-\pi < \theta \le \pi$, and this is called the **principal argument**.

Using polar coordinates the point A could be labelled with its polar coordinates $[r, \theta]$ as $\left[4, \frac{\pi}{6}\right]$. Note the use of squared brackets when using polar coordinates. This is to avoid confusion with Cartesian coordinates.

Thus $2\sqrt{3} + 2i = 4\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$.

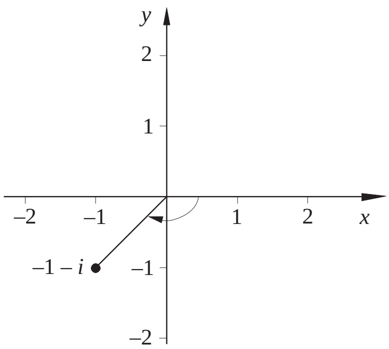
Important note: if you are expressing $a + ib$ in its polar form, where a and b are both positive, then the formula $\theta = \tan^{-1}\frac{b}{a}$ is quite sufficient. But in other cases you need to think about the position of $a + ib$ in the Argand diagram.

Example

Write $z = -1 - i$ in polar form.

Solution

Now $z = a + ib$ where $a = -1$ and $b = -1$ and in polar form the modulus of $z = |z| = r = \sqrt{1^2 + 1^2} = \sqrt{2}$ and the argument is $\frac{5\pi}{4}$ (or 225°): its principal value is $-\frac{3\pi}{4}$.



Hence $z = \left[\sqrt{2}, \frac{-3\pi}{4}\right]$ in polar coordinates. (The formula $\tan^{-1}\frac{b}{a}$ would have given you $\frac{\pi}{4}$.)

Activity 9

(a) Write the following numbers in $[r, \theta]$ form:

(i) $7 + 2i$ (ii) $3 - i$ (iii) $-4 + 6i$ (iv) $-\sqrt{3} - i$

(b) Write the following in $a + bi$ form:

(remember that the angles are in radians)

(i) $\left[3, \frac{\pi}{4}\right]$ (ii) $[5, \pi]$ (iii) $[6, 4.2]$

(iv) $\left[\sqrt{2}, \frac{-2\pi}{3}\right]$

3.5 Complex number algebra

You will now investigate the set of complex numbers in the modulus/argument form, $[r, \theta]$.

Suppose you wish to combine two complex numbers of the form

$$z_1 = [r_1, \theta_1] \qquad z_2 = [r_2, \theta_2]$$

Note that, in $a + bi$ form,

$$z_1 = r_1 \cos\theta_1 + i\, r_1 \sin\theta_1$$

and $$z_2 = r_2 \cos\theta_2 + i\, r_2 \sin\theta_2$$

So
$$\begin{aligned} z_1 z_2 &= \left(r_1 \cos\theta_1 + i\, r_1 \sin\theta_1\right)\left(r_2 \cos\theta_2 + i\, r_2 \sin\theta_2\right) \\ &= r_1 r_2 \left(\cos\theta_1 + i \sin\theta_1\right)\left(\cos\theta_2 + i \sin\theta_2\right) \\ &= r_1 r_2 \big[\left(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2\right) \\ &\qquad + \left(\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2\right) i\big]. \end{aligned}$$

Simplify the expressions in the brackets.

Using the formulae for angles,

$$\boxed{z_1 z_2 = r_1 r_2 \left[\cos\left(\theta_1 + \theta_2\right) + i \sin\left(\theta_1 + \theta_2\right)\right]}$$

or, in polar notation

$$z_1 z_2 = \left[r_1 r_2,\ \theta_1 + \theta_2\right].$$

For example, $[3, 0.5] \times [4, 0.3] = [12, 0.8]$.

That is, the first elements of the ordered pairs are **multiplied** and the second elements are **added**.

Activity 10

Given that $z_1 = [3, 0.7]$, $z_2 = [2, 1.2]$ and $z_3 = [4, -0.5]$,

(a) find $z_1 \times z_2$ and $z_1 \times z_3$

(b) show that $[1, 0] \times z_1 = z_1$

(c) (i) find a complex number $z=[r,\theta]$ such that $z \times z_2 = [1,0]$.

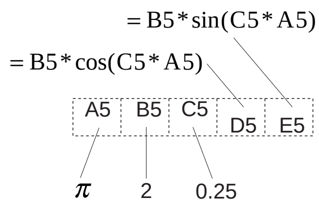
(ii) find a complex number $z=[r,\theta]$ such that $z \times z_3 = [1,0]$.

(d) for any complex number $[r,\theta]$ show that

$$\left[\frac{1}{r}, -\theta\right] \times [r,\theta] = [1,0] \qquad (r>0).$$

Activity 11

Use a spreadsheet package to plot numbers on an Argand diagram by entering numbers and formulae into cells A5 to E5 as shown opposite.



Cells D5 and E5 calculate the x and y coordinates respectively of the complex number whose modulus and argument are in cells B5 and C5 (the argument is entered as a multiple of π).

A second number can be entered in cells B6 and C6 and its (x, y) coordinates calculated by using appropriate formulae in cells D6 and E6.

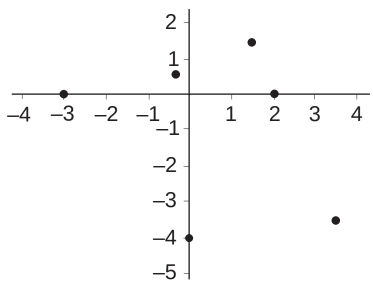
This can be repeated for further numbers (the spreadsheet facility 'FILL DOWN' is useful here).

Use the appropriate facility on your spreadsheet to plot the (x, y) values.

Label rows and columns if it makes it easier.

Experiment with different values of r and θ.

An example is shown in the graph opposite and the related spreadsheet below.



=B5*cos(C5*A5)

=B5*sin(C5*A5)

	A	B	C	D	E
1					
2					
3					
4					
5	3.141593	2	0.25	1.4142134399	1.4142136848
6	3.141593	3	1	-3.000000000	-1.039231E-6
7	3.141593	4	1.5	2.0784612E-6	-4.000000000
8	3.141593	0.6	0.666667	-0.300000664	0.5196148588
9	3.141593	2	2	2.0000000000	1.3856408E-6
10	3.141593	5	1.75	3.5355360492	-3.535531763
11					

Exercise 3B

1. Mark on an Argand diagram the points representing the following numbers:

   (a) 2 (b) $3i$ (c) $-i$ (d) $1+2i$ (e) $3-i$

   (f) $-2+3i$

2. The points A, B, C and D represent the numbers z_1, z_2, z_3 and z_4 and O is the origin.

   (a) If OABC is a parallelogram, and $z_1 = 1+i$, $z_2 = 4+5i$, find z_3.

   (b) Find z_2 and z_4 when ABCD is a square and

   (i) $z_1 = 2+i$, $z_3 = 6+7i$

   (ii) $z_1 = 6-2i$, $z_3 = 6i$

3. Find the modulus and argument of

   (a) $1-i$ (b) $1+\sqrt{3}i$ (c) $3-3i$ (d) $3+2i$

4. Show that

   (a) $|\bar{z}| = |z|$ (b) $\arg\bar{z} = -\arg z$

   and illustrate these results on an Argand diagram.

5. Find the modulus and argument of z_1, z_2, $z_1 z_2$ and $\dfrac{z_1}{z_2}$ when $z_1 = 1+i$ and $z_2 = \sqrt{3}+i$. What do you notice?

6. Write in the form $a+bi$

   (a) $\left[4, \dfrac{\pi}{3}\right]$ (b) $\left[5, \dfrac{\pi}{2}\right]$

   (c) $\left[3\sqrt{2}, -\dfrac{3\pi}{4}\right]$ (d) $[4, 13\pi]$

7. Write in polar form

   (a) $1+i$ (b) $-2+i$ (c) -5 (d) $4i$ (e) $3+4i$

   (f) $-3-4i$ (g) $3-4i$ (h) $-3+4i$

8. In this question, angles are in radians.

   (a) (i) Plot the following complex numbers on an Argand diagram and label them:

   $z_1 = [4, 0]$, $z_2 = \left[3, \dfrac{\pi}{2}\right]$, $z_3 = \left[2, \dfrac{-\pi}{2}\right]$

   $z_4 = \left[3, \dfrac{\pi}{3}\right]$, $z_5 = \left[2, \dfrac{5\pi}{3}\right]$

   (ii) Let the complex number $z = \left[1, \dfrac{\pi}{2}\right]$

   Calculate $z \times z_1$, $z \times z_2$, etc. and plot the points on the same diagram as in (i). What do you notice?

   (b) Repeat (a) (ii) using $z = \left[1, \dfrac{\pi}{3}\right]$

   (c) In general, what happens when a complex number is multiplied by $[1, \theta]$? Make up some examples to illustrate your answer.

   (d) Repeat (a) (ii) using $z = \left[0.5, \dfrac{\pi}{2}\right]$

   (e) In general, what happens when a complex number is multiplied by $\left[0.5, \dfrac{\pi}{2}\right]$? Make up some examples to illustrate your answer.

   (f) Repeat (e) for $\left[3, \dfrac{\pi}{3}\right]$

   (g) Describe what happens when a complex number is multiplied by $\left[3, \dfrac{\pi}{3}\right]$. Make up some examples to illustrate your answer.

3.6 De Moivre's theorem

An important theorem in complex numbers is named after the French mathematician, *Abraham de Moivre* (1667-1754). Although born in France, he came to England where he made the acquaintance of *Newton* and *Halley* and became a private teacher of Mathematics. He never obtained the university position he sought but he did produce a considerable amount of research, including his work on complex numbers.

The derivation of de Moivre's theorem now follows.

Consider the complex number $z=\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$.

Then $$z^2=\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\times\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$$

$$=\cos^2\frac{\pi}{3}-\sin^2\frac{\pi}{3}+2i\cos\frac{\pi}{3}\sin\frac{\pi}{3}$$

$$=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$$

or with the modulus/argument notation

$$z=\left[1,\frac{\pi}{3}\right]$$

and $$z^2=\left[1,\frac{\pi}{3}\right]\times\left[1,\frac{\pi}{3}\right]=\left[1,\frac{2\pi}{3}\right].$$

Remember that any complex number $z=x+yi$ can be written in the form of an ordered pair $[r,\theta]$ where $r=\sqrt{x^2+y^2}$ and $\theta=\tan^{-1}\left(\frac{y}{x}\right)$.

If the modulus of the number is 1, then $z=\cos\theta+i\sin\theta$

and $$z^2=(\cos\theta+i\sin\theta)^2$$

$$=\cos^2\theta-\sin^2\theta+2i\cos\theta\sin\theta$$

$$=\cos 2\theta+i\sin 2\theta$$

i.e. $$z^2=[1,\theta]^2=[1,2\theta].$$

Activity 12

(a) Use the principle that, with the usual notation,

$$[r_1,\theta_1]\times[r_2,\theta_2]=[r_1r_2,\theta_1+\theta_2]$$

to investigate $\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^n$ when $n=0,\ 1,\ 2,\ 3,\ ...,\ 12.$

(b) In the same way as in (a), investigate

$$\left(3\cos\frac{\pi}{6}+3i\sin\frac{\pi}{6}\right)^n$$

for $n=0,\ 1,\ 2,\ \ldots,\ 6.$

You should find from the last activity that

$$(\cos\theta+i\sin\theta)^n=\cos(n\theta)+i\sin(n\theta).$$

In $[r,\theta]$ form this is $[r,\theta]^n=\left[r^n,n\theta\right]$ and **de Moivre's theorem** states that this is true for any **rational number** n.

A more rigorous way of deriving de Moivre's theorem follows.

Activity 13

Show that $(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$ for $n=3$ and $n=4$.

Activity 14

Show that

$$(\cos k\theta+i\sin k\theta)(\cos\theta+i\sin\theta)=\cos(k+1)\theta+i\sin(k+1)\theta\ .$$

Hence show that if

$$(\cos\theta+i\sin\theta)^k=\cos k\theta+i\sin k\theta$$

then $(\cos\theta+i\sin\theta)^{k+1}=\cos((k+1)\theta)+i\sin((k+1)\theta).$

The principle of **mathematical induction** will be used to prove that $(\cos\theta+i\sin\theta)^n=\cos(n\theta)+i\sin(n\theta)$ for all **positive integers**.

Let $S(k)$ be the statement

$$'(\cos\theta+i\sin\theta)^k=\cos k\theta+i\sin k\theta'.$$

As $S(1)$ is true and you have shown in Activity 14 that $S(k)$ implies $S(k+1)$ then $S(2)$ is also true. But then (again by Activity 14) $S(3)$ is true. But then ... Hence S(n) is true for $n=1,2,3,\ldots.$ This is the principle of mathematical induction (which you meet more fully later in the book). So for all positive integers n,

$$\boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta}$$

If n is a **negative integer**, then let $m = -n$

$$(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m} = \frac{1}{(\cos\theta + i\sin\theta)^m}$$

where m is positive and, from the work above,

$$(\cos\theta + i\sin\theta)^m = (\cos m\theta + i\sin m\theta).$$

Therefore $\quad (\cos\theta + i\sin\theta)^n = \dfrac{1}{(\cos m\theta + i\sin m\theta)}$

Activity 15

Show that

$$\frac{1}{(\cos m\theta + i\sin m\theta)} = \cos m\theta - i\sin m\theta$$

and hence that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ when n is a negative integer.

Hint : multiply top and bottom by $(\cos m\theta - i\sin m\theta)$ and use the fact that $\sin(-A) = -\sin(A)$.

When n is a **rational number**, i.e. $n = \dfrac{p}{q}$ where p and q are integers, then as q is an integer

$$\left\{\cos\left(\frac{p}{q}\right)\theta + i\sin\left(\frac{p}{q}\right)\theta\right\}^q = (\cos p\theta + i\sin p\theta)$$

Since p is an integer

$$\cos p\,\theta + i\sin p\,\theta = (\cos\theta + i\sin\theta)^p,$$

and hence

$$\left\{\cos\left(\frac{p}{q}\right)\theta + i\sin\left(\frac{p}{q}\right)\theta\right\}^q = (\cos\theta + i\sin\theta)^p$$

Thus $$\left\{\cos\left(\frac{p}{q}\right)\theta + i\sin\left(\frac{p}{q}\right)\theta\right\} = (\cos\theta + i\sin\theta)^{\frac{p}{q}}$$

Therefore $\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$ for any rational number n and clearly this leads to

$$\boxed{(r(\cos\theta + i\sin\theta))^n = r^n(\cos n\theta + i\sin n\theta)}$$

3.7 Applications of de Moivre's theorem

There are many applications of de Moivre's theorem, including the proof of trigonometric identities.

Example

Prove that $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$.

Solution

By de Moivre's theorem:

$$\begin{aligned}\cos 3\theta + i\sin 3\theta &= (\cos\theta + i\sin\theta)^3\\ &= \cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3\\ &= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta\\ &= \cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta)\end{aligned}$$

Comparing **real parts** of the equation above you obtain

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$

Example

Simplify the following expression:

$$\frac{\cos 2\theta + i\sin 2\theta}{\cos 3\theta + i\sin 3\theta}$$

Solution

$$\frac{\cos 2\theta + i\sin 2\theta}{\cos 3\theta + i\sin 3\theta} = \frac{(\cos\theta + i\sin\theta)^2}{(\cos\theta + i\sin\theta)^3} = \frac{1}{(\cos\theta + i\sin\theta)^1}$$

$$= (\cos\theta + i\sin\theta)^{-1}$$

$$= \left(\cos(-\theta) + i\sin(-\theta)\right)$$

$$= \cos\theta - i\sin\theta$$

Exercise 3C

1. Use de Moivre's theorem to prove the trig identities:

   (a) $\sin 2\theta = 2\sin\theta\cos\theta$

   (b) $\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

2. If $z = \cos\theta + i\sin\theta$ then use de Moivre's theorem to show that:

   (a) $z + \dfrac{1}{z} = 2\cos\theta$ (b) $z^2 + \dfrac{1}{z^2} = 2\cos 2\theta$

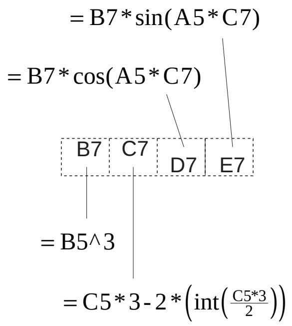
   (c) $z^n + \dfrac{1}{z^n} = 2\cos n\theta$

3. Simplify the following expressions:

   (a) $\dfrac{\cos 5\theta + i\sin 5\theta}{\cos 2\theta - i\sin 2\theta}$ (b) $\dfrac{\cos\theta - i\sin\theta}{\cos 4\theta - i\sin 4\theta}$

Activity 16

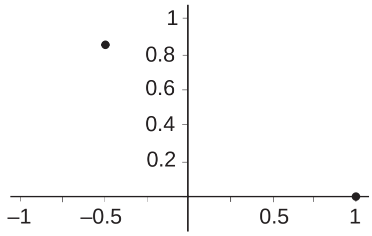
Make an educated guess at a complex solution to the equation $z^3 = 1$ and then use the facilities of the spreadsheet to raise it to the power 3 and plot it on the Argand diagram. If it is a solution of the equation then the resultant point will be plotted at distance 1 unit along the real axis. The initial spreadsheet layout from Activity 11 can be adapted. In addition, the cells shown opposite are required.



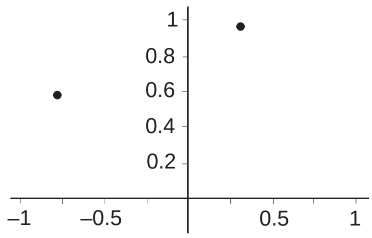
What does the long formula in cell C7 do? Is it strictly necessary in this context?

Below are two examples of the output from a spreadsheet using these cells – the first one is not a cube root of 1 but the second is.

	A	B	C	D	E
5	3.141593	1	0.6666667	-0.5	0.8666025
6					
7		1	2	1	4.14E-13

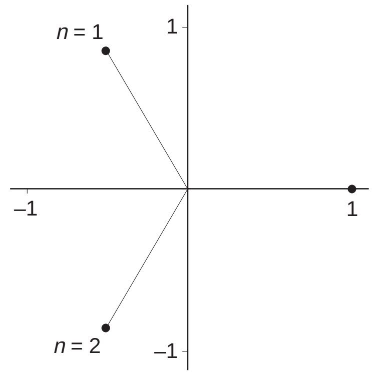


	A	B	C	D	E
5	3.141593	1	0.8	-0.80902	0.587785
6					
7		1	2.4	0.309017	0.951057



3.8 Solutions of $z^3 = 1$

Write down one solution of $z^3 = 1$.



De Moivre's theorem can be used to find all the solutions of $z^3 = 1$.

Let $\quad z = [r, \theta]$

then $\quad z^3 = [r, \theta]^3 = [r^3, 3\theta]$

and you can express 1 as $1 = [1, 2n\pi]$ where n is an integer.

Then $\quad [r^3, 3\theta] = [1, 2n\pi]$

Therefore $\quad r^3 = 1$ and $3\theta = 2n\pi$

i.e. $\quad r = 1$ and $\theta = \dfrac{2n\pi}{3}$

The solutions are then given by letting $n = 0, 1, 2, \ldots$

If $n = 0$, $\quad z_1 = [1, 0] = 1$

If $n = 1$,
$$z_2 = \left[1, \frac{2\pi}{3}\right] = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$$
$$= -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

If $n = 2$,
$$z_3 = \left[1, \frac{4\pi}{3}\right] = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$$
$$= -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

What happens if $n = 3, 4, \ldots$?

Activity 17 Cube roots of unity

Plot the three distinct cube roots of unity on an Argand diagram. What do you notice?

Activity 18

Use de Moivre's theorem to find all solutions to the following equations and plot the results on an Argand diagram.

(a) $z^4 = 1$ (b) $z^3 = 8$ (c) $z^3 = i$

3.9 Euler's theorem

You have probably already met the series expansion of e^x, namely

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

Also the series expansions for $\cos\theta$ and $\sin\theta$ are given by

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots$$

Activity 19

(a) For each of the following values of θ, use the series for e^x with x replaced by $i\theta$ to calculate (to 4 d.p.) the value of $e^{i\theta}$. (Write your answer in the form $a + bi$.)

(i) $\theta = 0$ (ii) $\theta = 1$ (iii) $\theta = 2$ (iv) $\theta = -0.4$

(b) Calculate $\cos\theta$ and $\sin\theta$ for each of the values in (*a*).

(c) Find a connection between the values of $e^{i\theta}$, $\cos\theta$ and $\sin\theta$ for each of the values of θ given in (a) and make up one other example to test your conjecture.

(d) To prove this for all values of θ, write down the series expansions of $e^{i\theta}$, $\cos\theta$ and $\sin\theta$ and show that

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

The previous activity has shown that

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

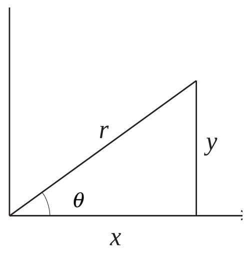
which is sometimes known as **Euler's theorem**.

It is an important result, and can be used to derive de Moivre's theorem in a simple way. If z is any complex number then in polar form

$$z = x + yi = r\cos\theta + r\,i\sin\theta$$

$$= r(\cos\theta + i\sin\theta)$$

$$= re^{i\theta}, \quad \text{using Euler's theorem.}$$



Thus $\quad z^n = \left(re^{i\theta}\right)^n = r^n e^{ni\theta} = r^n e^{i(n\theta)}$

or $\quad \left(r\cos\theta + ir\sin\theta\right)^n = r^n\left(\cos(n\theta) + i\sin(n\theta)\right)$

$$\Rightarrow \quad (\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

which is de Moivre's theoem.

What assumptions about complex number algebra have been made in the 'proof' above?

One interesting result can be obtained from Euler's theorem by putting $\theta = \pi$. This gives

$$e^{i\pi} = \cos\pi + i\sin\pi$$

$$= -1 + i \times 0.$$

So $\quad \boxed{e^{i\pi} + 1 = 0}$

This is often referred to as **Euler's equation**, since it connects the five most 'famous' numbers

$$0,\ 1,\ \pi,\ e,\ i$$

with a '+' and '=' sign!

Try substituting other values of θ in Euler's theorem and see what equation is derived.

3.10 Exponential form of a complex number

When a complex number z has modulus r, which must be non-negative, and argument θ, which is usually taken such that it satisfies $-\pi < \theta \le \pi$, you have already shown that it can be represented in the forms

(i) $r(\cos\theta + i\sin\theta)$

(ii) $[r,\ \theta]$

(iii) $re^{i\theta}$

Expression (iii) is referred to as the **exponential form** of a complex number.

Activity 20

Write each of the following complex numbers in the exponential form.

(a) $2\left(\cos\frac{\pi}{3} + i\cos\frac{\pi}{3}\right)$ (b) $\left[5,\ \frac{2\pi}{3}\right]$ (c) $1 - i\sqrt{3}$

3.11 Solving equations

You have already investigated the solutions of the equation $z^3 = 1$ and similar equations using a spreadsheet and by using de Moivre's theorem. A similar approach will now be used to solve more complicated equations.

Example

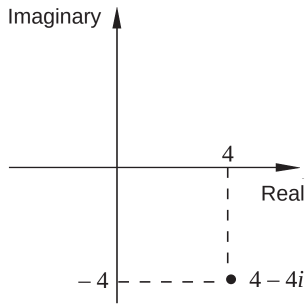
Write down the modulus and argument of the complex number $4 - 4i$.

Solve the equation $z^5 = 4 - 4i$, expressing your answers in the exponential form.

Solution

$$|4 - 4i| = \sqrt{\left\{4^2 + (-4)^2\right\}} = 4\sqrt{2}$$

As before it is often helpful to make a small sketch of an Argand diagram to locate the correct quadrant for the argument.



So $\arg(4-4i) = \dfrac{-\pi}{4}$

Therefore the complex number $4-4i$ can be expressed as

$$\left[4\sqrt{2},\ \frac{-\pi}{4}\right]$$

It is quite convenient to work using the polar form of a complex number when solving $z^5 = 4-4i$.

Let $z = [r,\ \theta]$, then $z^5 = \left[r^5,\ 5\theta\right]$.

So as to obtain all five roots of the equation, the argument is considered to be $2n\pi - \dfrac{\pi}{4}$ where n is an integer.

Equating the results

$$\left[r^5,\ 5\theta\right] = \left[4\sqrt{2},\ 2n\pi - \frac{\pi}{4}\right]$$

$$r^5 = 4\sqrt{2} \quad \Rightarrow \quad r = \sqrt{2}$$

$$5\theta = 2n\pi - \frac{\pi}{4} \quad \Rightarrow \quad \theta = (8n-1)\frac{\pi}{20}$$

Now choose the five appropriate values of n so that θ lies between $-\pi$ and π.

$$n=-2 \quad \Rightarrow \quad \theta = \frac{-17\pi}{20}$$

$$n=-1 \quad \Rightarrow \quad \theta = \frac{-9\pi}{20}$$

$$n=0 \quad \Rightarrow \quad \theta = \frac{-\pi}{20}$$

$$n=1 \quad \Rightarrow \quad \theta = \frac{7\pi}{20}$$

$$n=2 \quad \Rightarrow \quad \theta = \frac{15\pi}{20} \text{ or } \frac{3\pi}{4}$$

The solutions in exponential form are therefore

$$\sqrt{2}\,e^{-\frac{17\pi}{20}i},\ \sqrt{2}e^{-\frac{9\pi}{20}i},\ \sqrt{2}e^{-\frac{\pi}{20}},\ \sqrt{2}e^{\frac{7\pi}{20}i} \text{ and } \sqrt{2}e^{\frac{3\pi}{4}i}.$$

Activity 21

Show that $1+i$ is a root of the equation $z^4 = -4$ and find each of the other roots in the form $a+bi$ where a and b are real.

Plot the roots on an Argand diagram. By considering the diagonals, or otherwise, show that the points are at the vertices of a square. Calculate the area of the square.

Activity 22

Given that $k \neq 1$ and the roots of the equation $z^3 = k$ are α, β and γ, use the substitution $z = \dfrac{(x-2)}{(x+1)}$ to obtain the roots of the equation

$$(x-2)^3 = k(x+1)^3$$

Exercise 3D

1. By using de Moivre's theorem, find all solutions to the following equations, giving your answers in polar form. Plot each set of roots on an Argand diagram and comment on the symmetry.

   (a) $z^4 = 16$ (b) $z^3 = -27i$ (c) $z^5 = -1$

2. Find the cube roots of

   (a) $1+i$ (b) $2i-2$

   giving your answers in exponential form.

3. Using the answers from Question 1(a), determine the solutions of the equation

   $$(x+1)^4 = 16(x-1)^4$$

   giving your answers in the form $a+bi$.

4. Using the results from Question 1(b), solve the equation

   $$1+27i(x+1)^3 = 0$$

   giving your answers in the form $a+bi$.

5. Solve the equation $z^3 = i(z-1)^3$ giving your answers in the form $a+bi$.

   Plot the solutions on an Argand diagram and comment on your results.

6. Determine the four roots of the equation

   $$(z-2)^4 + (z+1)^4 = 0$$

   and plot them on an Argand diagram.

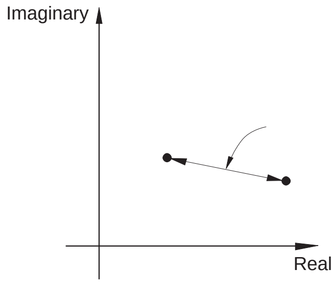
3.12 Loci in the complex plane

Suppose z is allowed to vary in such a way that $|z-1| = 2$. You could write $z = x+iy$ and obtain

$$\sqrt{\left\{(x-1)^2 + y^2\right\}} = 2$$

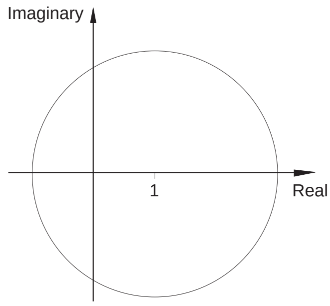
or $$(x-1)^2 + y^2 = 4$$

You can immediately identify this as the cartesian equation of a circle centre $(1,\ 0)$ and radius 2. In terms of the complex plane, the centre is $1+0i$.



This approach could be adopted for most problems and the exercise is simply one in algebra, lacking any geometrical feel for the locus.

Instead, if ω is a complex number, you can identify $|z-\omega|$ as the distance of z from the point represented by ω on the complex plane. The locus $|z-1|=2$ can be interpreted as the set of points that are 2 units from the point $1+0i$; in other words, a circle centre $1+0i$ and radius 2.



Activity 23

Illustrate the locus of z in the complex plane if z satisfies

(a) $|z-(3+2i)|=5$ (b) $|z-2+i|=|1+3i|$

(c) $|z+2i|=2$ (d) $|z-4|=0$

Activity 24

Describe the path of a point which moves in a fixed plane so that it is always the same distance from two fixed points A and B.

Illustrate the locus of z in the case when z satisfies

$$|z+3|=|z-4i|.$$

You would probably have had some difficulty in writing down a cartesian equation of the locus in Activity 24, even though you could describe the locus geometrically.

Activity 25

Describe the locus of z in the case where z moves in such a way that

$$|z|=|z+2-2i|.$$

Now try to write down the cartesian equation of this locus which should be a straight line.

By writing $z=x+iy$, try to obtain the same result algebraically.

Activity 26

Investigate the locus of P when P moves in the complex plane and represents the complex number z which satisfies

$$|z+1| = k|z-1|$$

for different values of the real number k.

Why does $k=1$ have to be treated as a special case?

Example

The point P represents the complex number z on an Argand diagram. Describe the locus geometrically and obtain a cartesian equation for the locus in the cases

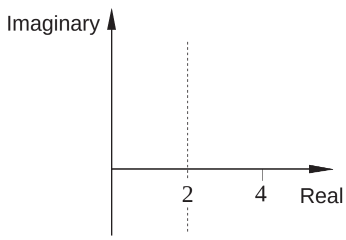
(a) $|z| = |z-4|$

(b) $|z| + |z-4| = 6$

(c) $|z| = 2|z-4|$

Solution

(a) From your work in Activity 25, you should recognise this as a straight line. In fact, it is the mediator, or perpendicular bisector, of the line segment joining the origin to the point $4+0i$.



It should be immediately obvious that its cartesian equation is $x=2$; however, writing

$$z = x+iy$$

$$|z| = |x+iy| = |x-4+iy|$$

Squaring both sides gives

$$x^2+y^2 = (x-4)^2+y^2$$

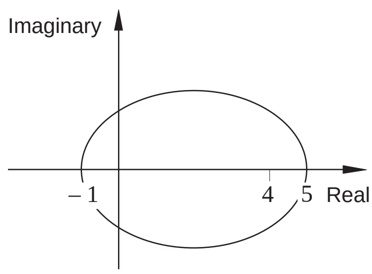
leading to

$$0 = -8x+16$$

or $\quad x = 2$.

(b) You may be aware of a curve that is traced out when the sum of the distances from two fixed points is constant. You could try using a piece of string with its ends fastened to two fixed points. The curve is called an ellipse.

A sketch of the locus is shown opposite.



You can obtain a cartesian equation by putting $z = x + iy$

$$|x+iy| + |x-4+iy| = 6$$

So $$\sqrt{x^2+y^2} + \sqrt{\left((x-4)^2+y^2\right)} = 6$$

$$\Rightarrow \quad (x-4)^2 + y^2 = \left[6 - \left(x^2+y^2\right)\right]^2$$

$$x^2 - 8x + 16 + y^2 = 36 - 12\sqrt{\left(x^2+y^2\right)} + x^2 + y^2$$

$$12\sqrt{\left(x^2+y^2\right)} = 20 + 8x$$

$$3\sqrt{\left(x^2+y^2\right)} = 5 + 2x$$

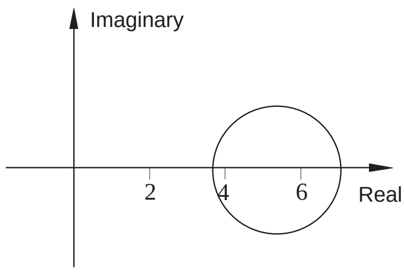
$$9\left(x^2+y^2\right) = 25 + 20x + 4x^2$$

$$5x^2 - 20x + 9y^2 = 25$$

$$5(x-2)^2 + 9y^2 = 45$$

$$\frac{(x-2)^2}{9} + \frac{y^2}{5} = 1$$

(c) You should have discovered in Activity 26 that the locus will be a circle when the relationship is of this form. It is called the **circle of Apollonius**.



You could possibly sketch the locus without finding the cartesian equation.

Let $$z = x + iy$$

$$|x+iy| = 2|x-4+iy|$$

$$\sqrt{\left(x^2+y^2\right)} = 2\sqrt{\left((x-4)^2+y^2\right)}$$

$$x^2 + y^2 = 4\left(x^2 - 8x + 16 + y^2\right)$$

$$0 = 3x^2 + 3y^2 - 32x + 64$$

In order to find the centre and radius you can complete the square

$$x^2 + y^2 - \frac{32}{3}x + \frac{64}{3} = 0$$

$$\left(x - \frac{16}{3}\right)^2 + y^2 = \frac{256}{9} - \frac{64}{3} = \frac{64}{9}$$

Centre of circle is at $\frac{16}{3} + 0i$ and radius is $\frac{8}{3}$.

Activity 27

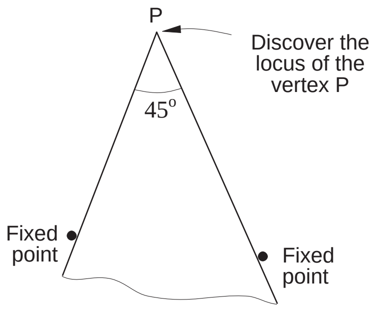
By recognising the locus

$$|z-2| = 3|z-10|$$

as the circle of Apollonius, use the idea of simple ratios to determine the coordinates of the centre and the radius of the circle.

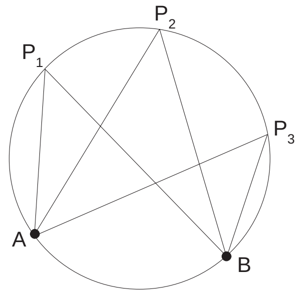
Check your answer by finding the cartesian equation of the circle.

Activity 28



By folding a piece of paper, create an angle of 45° and cut it out. Now mark two fixed points on a piece of paper and explore the locus of the vertex as you keep the two sides of the cut-out in contact with the fixed points as shown.

You should find that P moves on the arc of a circle.



Alternatively, when you have a circle and two fixed points A and B, if you choose a sequence of points $P_1,\ P_2, P_3, \ldots$ on the circumference, what do you notice about the angles $AP_1B,\ AP_2B,\ AP_3B,$ etc.?

This is an example of the constant angle locus.

Example

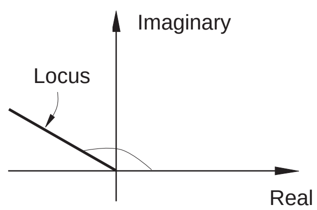
The point P represents z in the complex plane. Find the locus of P in each of the cases below when z satisfies

(a) $\arg z = \dfrac{5\pi}{6}$

(b) $\arg(z - 2 + 3i) = \dfrac{-\pi}{4}$

(c) $\arg\left(\dfrac{z-1}{z+1}\right) = \dfrac{\pi}{4}$

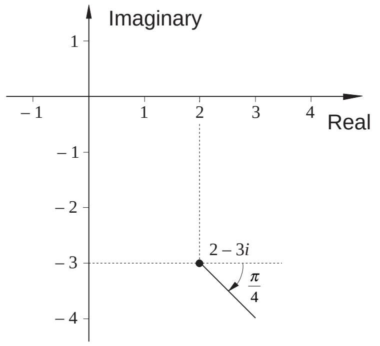
Solution



(a) The locus is a half-line starting at the origin making an angle $\dfrac{5}{6}\pi$ with the real axis.

(b) The half-line to be considered here is one which starts at the point $2-3i$.

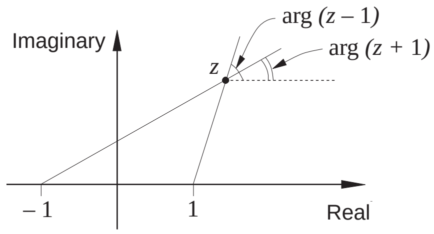
It makes an angle of $\dfrac{\pi}{4}$ below the real axis as shown opposite.



(c) You need to make use of the fact that

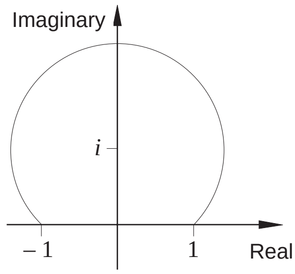
$$\arg\left(\frac{z-1}{z+1}\right)=\arg(z-1)-\arg(z+1)$$

One possible solution for z is shown in the second diagram opposite.



By the results of Activity 28 you can see that the locus of z is the major arc of a circle passing through $1+0i$ and $-1+0i$.

Since the angle at the centre of the circle is twice that on the circumference, it can be seen that the centre of the circle is at $0+i$ and hence the radius of the circle is $\sqrt{2}$.



The problem can be tackled algebraically but there are difficulties that can creep in by assuming

$$\arg(x+iy)=\tan^{-1}\left(\frac{y}{x}\right)$$

Nevertheless, you can obtain the cartesian equation of the full circle of which the locus is only part.

Let $\qquad z=x+iy$

$$\arg\left(\frac{z-1}{z+1}\right) = \arg\left(\frac{x-1+iy}{x+1+iy}\right)$$

$$= \arg\left[\frac{\{(x-1)+iy\}\{(x+1)-iy\}}{(x+1)^2+y^2}\right]$$

$$= \arg\left[\frac{\left(x^2-1+y^2\right)+2iy}{(x+1)^2+y^2}\right]=\frac{\pi}{4}$$

Taking tangents of both sides

$$\frac{2y}{x^2-1+y^2}=1$$

$$\Rightarrow \qquad x^2+y^2-2y=1$$

$$x^2+(y-1)^2=2$$

from which we see the centre is $0+i$ and the radius is $\sqrt{2}$.

Note: this approach does not indicate whether the locus is the major or minor arc of the circle and so the first approach is recommended.

Exercise 3E

1. Sketch the locus of z described by

   (a) $|z+3-4i|=5$ (b) $|z+2|=|z-5+i|$

   (c) $|z+3i|=3|z-i|$ (d) $|z-2|+|z-3+i|=0$

2. Describe geometrically and obtain a cartesian equation for the locus of z in each of the following cases.

   (a) $|z-3|^2=100$ (b) $|z-1|^2+|z-4|^2=9$

   (c) $|z-1|+|z-4|=5$ (d) $|z-1|-|z-4|=1$

3. Describe geometrically and sketch the region on the complex plane for which

   (a) $2<|z-3+i|\le 5$ (b) $\frac{-\pi}{4}\le \arg(z-2i)\le \frac{\pi}{3}$

4. Sketch the loci for which

   (a) $\arg\left(\frac{z+1}{z-i}\right)=\frac{3\pi}{2}$ (b) $\arg(z-2)^3=\frac{\pi}{2}$

   (c) $\arg(z+2)-\arg(z-3)=\frac{\pi}{3}$

   (d) $\arg\left(\frac{z-5+7i}{z+1+i}\right)=\frac{\pi}{2}$

3.13 Miscellaneous Exercises

1. Find the modulus and argument of the complex numbers

   $z_1=1+i$ and $z_2=1-\sqrt{3}i.$

   Hence find in the form $z=[r,\ \theta]$ where $-\pi<\theta\le\pi$ and $r>0$, the complex numbers

   (a) z_1z_2 (b) $\frac{z_1}{z_2}$ (c) $\frac{z_2}{z_1}$

   (d) $z_1{}^2$ (e) $z_2{}^3$ (f) $\frac{z_1{}^2}{z_2{}^4}$

2. Express the numbers $1,\ 3i,\ -4,\ z=2+\sqrt{5}i$ in the $[r,\ \theta]$ form. Hence express in the $[r,\ \theta]$ form

   (a) $\frac{1}{z}$ (b) $3zi$ (c) $\frac{z}{3i}$

   (d) $-4z$ (e) $\frac{-4}{z}$

3. Find $\sqrt{3}+i$ in the $[r,\ \theta]$ form. Hence find

   (a) $\left(\sqrt{3}+i\right)^3$ (b) $\left(\sqrt{3}+i\right)^8$

   in the form $a+bi$.

   (c) Find the least value of the positive integer n for which $\left(\sqrt{3}+i\right)^n$ is

   (i) purely real

   (ii) purely imaginary.

4. Find in the form $a+bi$

   (a) $\left(1+\sqrt{3}i\right)^5$ (b) $\left(\sqrt{3}-i\right)^{10}$ (c) $(1-i)^7$

   by making use of de Moivre's theorem.

5. Simplify $(1+i)^{10}-(1-i)^{10}$.

   Given that n is a positive integer, show that

   $(1+i)^{4n}-(1-i)^{4n}=0.$

6. Given that $z=\frac{\sqrt{3}}{2}+\frac{1}{2}i$, simplify $z^2,\ z^3,\ z^4$ and illustrate each of these numbers as points on an Argand diagram.

7. Show that the three roots of $z^3=1$ can be expressed in the form $1,\ \omega,\ \omega^2$.

   Hence show that $1+\omega+\omega^2=0$.

   Using this relation and the fact that $\omega^3=1$, simplify the following

   (a) $(1+\omega)^7$ (b) $(1-\omega)\left(1-\omega^2\right)$

   (c) $\frac{\omega^5}{1+\omega}$ (d) $\left(1-\omega+\omega^2\right)^4$

   (e) $\left(\omega-\omega^2\right)^5$ (f) $\frac{\left(1+\omega^2\right)(1-\omega)}{(1+\omega)}$

8. The roots of the equation $z^2+4z+29=0$ are z_1 and z_2. Show that $|z_1|=|z_2|$ and calculate, in degrees, the argument of z_1 and the argument of z_2.

   In an Argand diagram, O is the origin and z_1 and z_2 are represented by the points P and Q.

   Calculate the radius of the circle passing throught the points O, P and Q. (AEB)

9. Sketch on an Argand diagram the loci given by

$$|z-1-2i|=5$$

$$|z-5+i|=|z+3-5i|$$

Show that these loci intersect at the point z_1 where $z_1=-2-2i$, and at a second point z_2. Find z_2 in the form $a+bi$, where a and b are real. Express z_1 in the form $r(\cos\alpha+i\sin\alpha)$ where $r>0$ and $-\pi<\alpha\le\pi$, giving the value of r and the value of α. Show that z_1 is a root of the equation $z^4+64=0$.

Express z^4+64 in the form

$$\left(z^2+Az+B\right)\left(z^2+Cz+D\right)$$

where A, B, C and D are real, and find these numbers. (AEB)

10. (a) Find the modulus and argument of the complex number $\dfrac{\sqrt{3}+i}{1+i\sqrt{3}}$ giving the argument in radians between $-\pi$ and π.

(b) Find the value of the real number λ in the case when $\dfrac{\sqrt{3}+i\lambda}{1+i\sqrt{3}}$ is real. (AEB)

11. The complex number $u=-10+9i$

(a) Show the complex number u on an Argand diagram.

(b) Giving your answer to the nearest degree, calculate the argument of u.

(c) Find the complex number v which satisfies the equation

$$uv=-11+28i.$$

(d) Verify that $|u+v|=8\sqrt{2}$. (AEB)

12. (a) The complex number z satisfies the equation $|z+1|=\sqrt{2}|z-1|$. The point P represents z on an Argand diagram. Show that the locus of P is a circle with its centre on the real axis, and find its radius.

(b) Find the four roots of the equation

$$(z+1)^4+4(z-1)^4=0,$$

expressing the roots z_1, z_2, z_3 and z_4 in the form $a+bi$.

Show that the points on an Argand diagram representing z_1, z_2, z_3 and z_4 are the vertices of a trapezium and calculate its area. (AEB)

13. Let z be the complex number $-1+\sqrt{3}i$.

(a) Express z^2 in the form $a+bi$.

(b) Find the value of the real number p such that z^2+pz is real.

(c) Find the value of the real number q such that

$$\operatorname{Arg}\left(z^2+qz\right)=\frac{5\pi}{6}.$$

(AEB)

14. Use the method of mathematical induction to prove that

$$(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta,$$

where n is a positive integer.

Deduce that the result is also true when n is a negative integer.

Show that

$$2\cos n\theta=z^n+z^{-n},$$

where $z=\cos\theta+i\sin\theta$.

By considering $\left(z+z^{-1}\right)^4$, show that

$$\cos^4\theta\equiv\frac{1}{8}(\cos 4\theta+4\cos 2\theta+3).$$

Hence evaluate $\displaystyle\int_0^{\frac{\pi}{6}}\cos^4 2\theta\,d\theta$. (AEB)

15. You are given the complex number

$$\omega=\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}.$$

(a) Write down the value of ω^5 and prove that

$$1+\omega+\omega^2+\omega^3+\omega^4=0.$$

Simplify $\left(\omega+\omega^4\right)\left(\omega^2+\omega^3\right)$.

Form a quadratic equation with integer coefficients having roots

$$\left(\omega+\omega^4\right)\text{ and }\left(\omega^2+\omega^3\right)$$

and hence prove that

$$\cos\frac{2\pi}{5}=\frac{-1+\sqrt{5}}{4}.$$

(b) In an Argand diagram the point P is represented by the complex number z.

Sketch and describe geometrically in each case, the locus of the point P when

(i) $|z-\omega|=|z-1|$

(ii) $\arg\left(\dfrac{z-\omega}{z-1}\right)=\dfrac{\pi}{5}$.

(AEB)

16. (a) Use de Moivre's theorem to show that

$$\left(\sqrt{3}-i\right)^n = 2^n\left(\cos\frac{n\pi}{6} - i\sin\frac{n\pi}{6}\right),$$

where n is an integer.

(i) Find the least positive integer m for which $\left(\sqrt{3}-i\right)^m$ is real and positive.

(ii) Given that $\left(\sqrt{3}-i\right)$ is a root of the equation $z^9 + 16(1+i)z^3 + a + ib = 0$,

find the values of the real constants a and b.

(b) The point P represents a complex number z on an Argand diagram and

$$\left|z-\omega^6\right| = 3\left|z-\omega^3\right|,$$

where $\omega = \sqrt{3}-i$.

Show that the locus of P is a circle and find its radius and the complex number represented by its centre.

(AEB)