Dot product in dimension 3
(fr : produit scalaire dans l'espace)

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1. Definitions and properties

> ***Definition :***
> ***Let $\vec{u}$ and $\vec{v}$ being two 3D vectors. There exists a plane P containing the points A, B et C such that $\overrightarrow{AB} = \vec{u}$ and $\overrightarrow{AC} = \vec{v}$.***
> ***The dot product $\vec{u}\cdot\vec{v}$ in the 3D space is equal to the dot product $\vec{u}\cdot\vec{v}$ calculated in the plane P.***

Note : All the properties seen in the plane (in 1°) are thus still valid :

- ✓ With orthogonal projections : $\overrightarrow{AB}\cdot\overrightarrow{AC} = \overrightarrow{AB}\cdot\overrightarrow{AH}$ where H is the orthogonal projection of C on (AB)
- ✓ With angles and lengths : $\vec{u}\cdot\vec{v} = \|\vec{u}\|\times\|\vec{v}\|\times\cos(\vec{u};\vec{v})$ $\qquad$ $\overrightarrow{AB}\cdot\overrightarrow{AC} = AB\times AC\times\cos BAC$

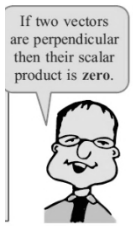
  $\vec{u}\cdot\vec{v} = \frac{1}{2}\left(\|\vec{u}\|^2 + \|\vec{v}\|^2 - \|\vec{u}-\vec{v}\|^2\right)$ $\qquad$ $\overrightarrow{AB}\cdot\overrightarrow{AC} = \frac{1}{2}\left(AB^2 + AC^2 - BC^2\right)$
- ✓ $\|\vec{u}\| = \sqrt{\vec{u}^2}$

> ***Properties :***
> ***In a 3D orthonormal coordinate system, for any vectors $\vec{u}(x;y;z)$ and $\vec{v}(x';y';z')$:***
> $\|\vec{u}\| = \sqrt{x^2 + y^2 + z^2}$ $\qquad$ $\vec{u}\cdot\vec{v} = xx' + yy' + zz'$
> ***All the calculation rules we have just revised remain valid in 3D.***

Proof :

$\vec{u}\cdot\vec{v} = \frac{1}{2}\left(\|\vec{u}\|^2 + \|\vec{v}\|^2 - \|\vec{u}-\vec{v}\|^2\right) = \ldots.$

> ***Properties : for any vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ and real number k :***
> $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$ $\qquad$ $(k\vec{u})\cdot\vec{v} = k(\vec{u}\cdot\vec{v})$ $\qquad$ $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$
> $\vec{u}\cdot\vec{u} = \vec{u}^2 = \|\vec{u}\|^2$
> $\vec{u}\cdot\vec{v} = 0 \Leftrightarrow$ ***the vectors are orthogonal***
> ***Special identities :*** $(\vec{u}\pm\vec{v})^2 = \vec{u}^2 \pm 2\vec{u}\cdot\vec{v} + \vec{v}^2$



Note :
Two lines with direction vectors $\vec{u}$ and $\vec{v}$ are orthogonal if and only if $\vec{u}$ and $\vec{v}$ are orthogonal.

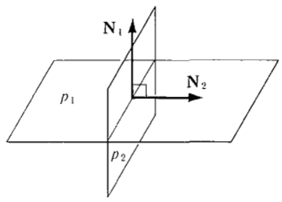
2. Orthogonality in space

> ***Property :***
> ***A line D with direction vector $\vec{u}$ is perpendicular to a plane (P) if there exists in (P) two non-collinear vectors orthogonal to $\vec{u}$.***

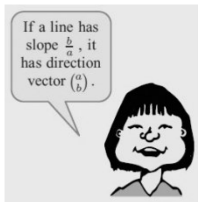
Definition and property : Normal vector to a plane
Let A be a point and $\vec{n}$ a non-null vector.
The set of points M satisfying $\overrightarrow{AM} \cdot \vec{n} = 0$ is the plane (P) passing through A and perpendicular to any line having $\vec{n}$ for direction vector.
$\vec{n}$ is then called <u>normal vector</u> to P.

special case : The set of points equidistant of two points A and B is a plane with normal vector $\overrightarrow{AB}$, called median plane (fr: plan médiateur) of [AB].

Property :
Two planes are perpendicular if their normal vectors are orthogonal.



Property : Cartesian equation of a plane
In a 3D orthonormal coordinate system :
1. Any plane (P) with normal vector $\vec{n}(a;b;c)$ has an Cartesian equation in the form $ax+by+cz+d=0$ (a, b, and c being real numbers)
2. Any equation in the form $ax+by+cz+d=0$ is the one of a plane with normal vector $\vec{n}(a;b;c)$.



Proof

$M \in (P) \Leftrightarrow \overrightarrow{AM}.\vec{n} = 0$

Example : Cartesian equation of (P) passing through $A(2;-1;0)$ and with normal vector is $\vec{n}(-1;2;3)$

Bullet points of the chapter

- ✓ Working out if a vector is normal to a plane
- ✓ Using the plane characterization by its equation ax+by+cz+d=0
- ✓ Finding a plane equation knowing a point and a normal vector
- ✓ Finding a normal vector to a plane given by its Cartesian equation
- ✓ Being able to prove that a line is orthogonal to a plane if and only if, it is orthogonal to two intersecting lines of this plane
- ✓ Using both parametric characterization and Cartesian equation to study the intersection of a line and a plane or the relative position of two planes