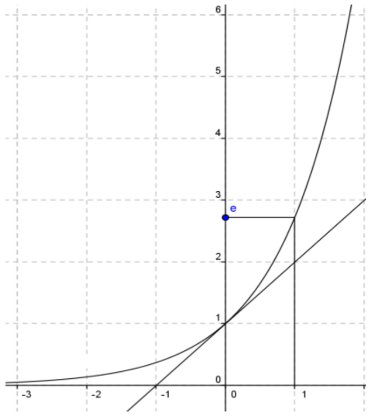
THE NATURAL LOGARITHM FUNCTION
(FR: FONCTION LOGARITHME NEPERIEN)

Ecole Internationale PACA à Manosque Tle S

1. DEFINITION AND FIRST PROPERTIES

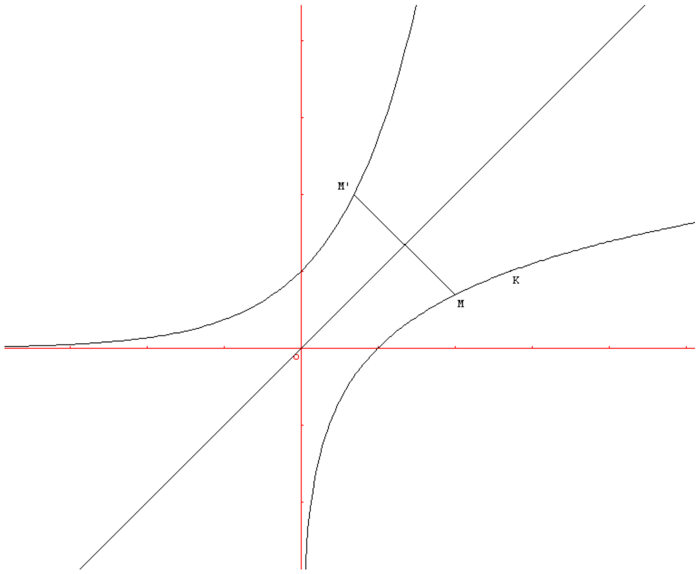


The exponential function $y \mapsto e^y$ is continuous and strictly increasing on $\mathbb{R}$. For any x in $\mathbb{R}^{+*}$, there thus exists a unique (thanks to the intermediate value theorem) real number y such that $e^y = x$.
We say that the exponential function realizes a <u>bijection</u> from $\mathbb{R}$ into $\mathbb{R}^{+*}$.

Definition :
The natural logarithm function (fr : fonction logarithme Népérien [1]), denoted ln is defined on $]0;+\infty[$ ***by :***
$x \mapsto y = \ln(x)$ ***with*** $e^y = x$

Note :

- we denote $\ln(x)$ or $\ln x$
- ***ln*** key on calculators : $\ln 5 \approx$ $\ln 100 \approx$



Properties :
1. For any $x > 0$ ***,*** $e^y = x \Leftrightarrow y = \ln x$ ***we say that the functions exponential et natural logarithm are <u>inverse</u> (fr: <u>réciproques)</u>***
2. Their graph are <u>symmetrical about the line with equation</u> $y = x$
3. For any $x > 0$ ***,*** $e^{\ln x} = x$
4. For any $x \in \mathbb{R}$ ***,*** $\ln e^x = x$
5. $\ln 1 = 0$ ***and*** $\ln e = 1$

Proof (2) :
$M(x;y) \in C_{\ln} \Leftrightarrow y = \ln x \Leftrightarrow x = e^y \Leftrightarrow M'(y;x) \in C_{\exp}$ M' is symmetrical to M about the line with equation $y = x$

2. STUDY OF THE LN FUNCTION

2.1 CARACTERISTIC PROPERTY.

Property :
For any real numbers a and b in $]0;+\infty[$: $\ln(ab) = \ln a + \ln b$

Proof :

[1] Named after John Napier of Merchiston (1550 – 4 April 1617) – also signed as Neper – named Marvellous Merchiston, was a Scottish mathematician, physicist, astronomer & astrologer, who as invented them

This property is caracteristic :

The only differentiable functions f on $]0;+\infty[$ satisfying $f(ab)=f(a)+f(b)$ are the ones in the form $f(x)=k\times\ln(x)$ with $k\in\mathbb{R}$

Note : For any strictly positive real number y, we can define the function $\log_y(x)$ (logarithm to base y of x) by : $\log_y(x)=\dfrac{\ln x}{\ln y}$. One important special case is the decimal logarithm (denoted $\log_{10} x$ or simply $\log x$) for which $\log 10^k = k$ for any k in $\mathbb{Z}$.

2.2 Calculation Properties

Properties : For any real numbers a and b in $]0;+\infty[$ and n integer number,

$$\ln\left(\frac{1}{a}\right)=-\ln a \qquad \ln\left(\frac{a}{b}\right)=\ln a-\ln b \qquad \ln\left(a^n\right)=n\ln a \qquad \ln\left(\sqrt{a}\right)=\frac{1}{2}\ln a$$

Proofs : (to be known, they all come from the characteristic property)

2.3 Limits

Property :

$$\lim_{x\to+\infty}\ln(x)=+\infty \qquad \lim_{x\to0^+}\ln(x)=-\infty \qquad \lim_{x\to+\infty}\frac{\ln x}{x}=0$$

Proof : Deduced from the exp function by symmetry about the line with equation $y=x$

2.4 Continuity and differentiability.

Property : on $]0;+\infty[$,

1. The ln function is continuous,

2. The ln function is differentiable and $\ln'(x)=\dfrac{1}{x}$

Proof :

1. Admited
2. At a number a in $]0;+\infty[$, does $\lim_{h\to0}\left(\dfrac{\ln(a+h)-\ln a}{h}\right)$ exist ?

We denote $k=\ln(a+h)$, $b=\ln a$. Then :

- ✓ The ln function being continuous, $\lim_{h\to0}\ln(a+h)=\ln(a)$, thus $\lim_{h\to0}k=b$
- ✓ 2. $h=a+h-a=e^k-e^b$ and $\left(\dfrac{\ln(a+h)-\ln a}{h}\right)$ can be written $\left(\dfrac{k-b}{e^k-e^b}\right)$ that is $\dfrac{1}{\dfrac{e^k-e^b}{k-b}}$.

We thus have $\lim_{h\to 0}\left(\frac{\ln(a+h)-\ln a}{h}\right)=\lim_{k\to b}\frac{k-b}{e^k-e^b}=\lim_{k\to b}\frac{1}{\frac{e^k-e^b}{k-b}}=\frac{1}{e^b}$ (since the exp function is differentiable on $\mathbb{R}$ and specially at b) . But $\frac{1}{e^b}=\frac{1}{a}$, proving that $\ln'(a)=\frac{1}{a}$ for any $a>0$.

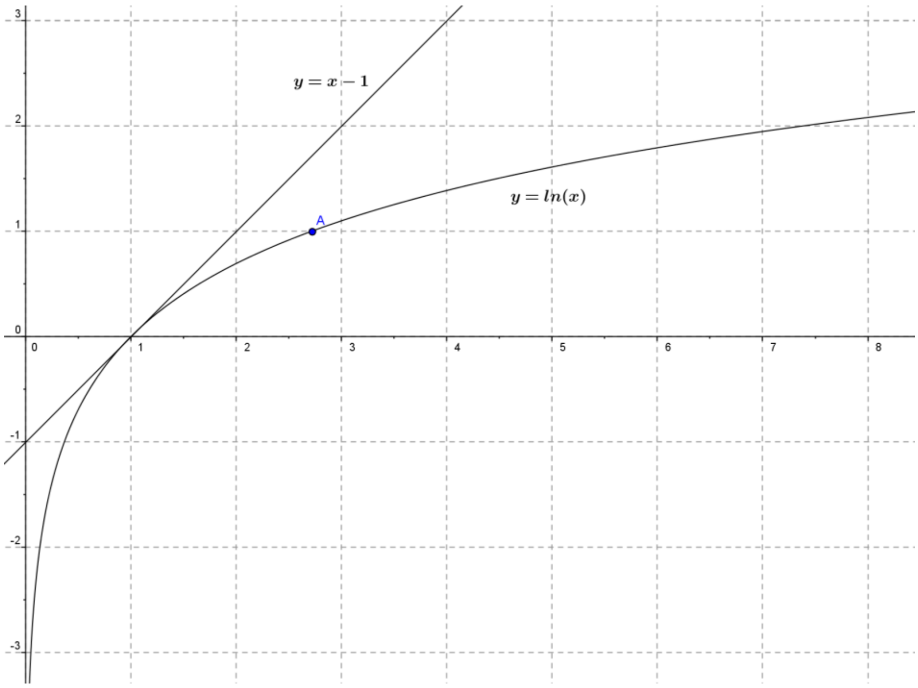
Note :

Equation of the tangent at 1 :

$\lim_{x\to 0}\frac{\ln(1+x)}{x}=?$

2.5 VARIATIONS.

Property :
The ln function is strictly increasing on $]0;+\infty[$



Proof :

$\ln'(x)=\frac{1}{x}>0$ on $]0;+\infty[$

Direct consequences (Important for equations or inequations solving) : for any positive real numbers a and b :

* $a<b \Leftrightarrow \ln a<\ln b$
* $\ln a<0 \Leftrightarrow 0<a<1$
* $\ln a>0 \Leftrightarrow a>1$

Example:

Solve: a) $\ln(2x-1)=\ln(x-2)$

b) $\ln[(x+3)(x-2)]<\ln 6$

Find the biggest natural numbers n satisfying: $(0.8)^n \geq 10^{-9}$

3. DERIVATIVE OF LN(U)

Property : u being a differentiable and strictly positive function on an interval, then the function $\ln \circ u = \ln(u)$ exists and is differentiable and $\ln'(u) = \frac{u'}{u}$

Example : $f(x) = \ln(x^2 - 9)$

Note : u and $\ln(u)$ have always the same variations.

Key points of the chapter

Using standard function to solve equations a[...]
inequalities