Sequences (part I) :
Revisions and Proof by Induction

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1. Revisions.

1.1 Variations.

- ✓ u increasing $\Leftrightarrow u_n \leq u_{n+1}$ for any n in $\mathbb{N}$
- ✓ u decreasing $\Leftrightarrow u_n \geq u_{n+1}$ for any n in $\mathbb{N}$
- ✓ to work out the variations of a given sequence u, we can analyse the sign of $u_{n+1} - u_n$
- ✓ if $u_n = f(n)$ and f is increasing on $[0;+\infty[$, then u is increasing (idem for decreasing)
- ✓ if u has all its terms positive then, if $\frac{u_{n+1}}{u_n} \geq 1$, u is increasing
- ✓ if $\frac{u_{n+1}}{u_n} \leq 1$, u is decreasing
- ✓ beware, if the sequence is defined by recurrence ($u_n = f(u_{n-1})$), its variations are depending on f and u_0.
- ✓ if u is increasing or decreasing, we say it is monotonic.

1.2 Bounded sequences.

Definitions :
**** a sequence u is bounded above by M (respectively bounded below by m) if*** $\forall n \in \mathbb{N};\ u_n \leq M$ ***(respectively*** $\forall n \in \mathbb{N};\ u_n \geq m$ ***) (we say that M is an upper bound of u or m is a lower bound of u)***
**** a sequence which is either bounded above or bounded below is said bounded***

1.3 Arithmetic sequences.

Definition :
(u_n) *arithmetic* $\Leftrightarrow \forall n \in \mathbb{N};\ u_{n+1} = u_n + r$ ***r being called the common difference of the sequence***

Note : We often prove a sequence is arithmetic showing that the difference $u_{n+1} - u_n$ is a constant number.

Properties :
**** expression of*** u_n ***in terms of n :*** $\boxed{u_n = u_p + (n-p)r}$. ***Particular case :*** $\boxed{u_n = u_0 + nr}$
**** sum of consecutive terms :*** $\sum_{p=m}^{p=m'} u_p = (m'-m+1)\frac{u_m + u_{m'}}{2}$. ***Special case :*** $\boxed{\sum_{p=0}^{p=n} u_p = (n+1)\frac{u_0 + u_n}{2}}$
(literally : « nb of terms times arithmetic mean of the first one and the last one »)

1.4 Geometric sequences.

Definition :
(u_n) *geometric* $\Leftrightarrow \forall n \in \mathbb{N};\quad u_{n+1} = q \times u_n$ ***q being called the common ratio of the sequence***

Note : We often prove a sequence is geometric showing that the ratio $\frac{u_{n+1}}{u_n}$ is a constant number.

Properties :
**** expression of*** u_n ***in terms of n :*** $\boxed{u_n = u_p \times q^{n-p}}$***. Particular case :*** $\boxed{u_n = u_0 \times q^n}$

**** sum of consecutive terms (if*** $q \neq 1$***) :*** $\sum_{p=m}^{p=m'} u_p = \frac{u_{m'+1} - u_m}{q-1} = u_m \frac{q^{m'-m+1} - 1}{q-1}$ ***. Special case :*** $\boxed{\sum_{p=0}^{p=n} u_p = \frac{u_{n+1} - u_0}{q-1} = u_0 \frac{q^{n+1} - 1}{q-1}}$ ***(literally : « first term times q to the power of (nb of terms) minus 1 over (q minus 1) »)***

2. Proof by induction.

2.1 Introduction : "two-colours map theorem"

a. Draw up opposite 10 lines, at random, the way you like.
b. Imagine the area between lines are countries. Colour the map with only 2 different colours, so that 2 countries having a common boundary are not in the same colour.

The two-color map theorem states it is always possible.
How can we prove it ?

c. Add a line on your figure. Could you find a rule to colour now the map?

d. What would then be the proof ?

2.2 Principle of induction

Let's call $P(n)$ an assertion which depends on a natural integer number n and n_0 a natural integer number . To prove that $P(n)$ is true for all integer number $n \geq n_0$, it is enough to prove that :

- ✓ $P(n_0)$ is true
- ✓ $P(n)$ is hereditary, that's $\left(P(n)\, true \Rightarrow P(n+1)\, true\right)$ for any integer number $n \geq n_0$

Note : * The first step is called initialisation
* Beware of verifying both initialisation and heredity (see ex 20)

Examples : Prove by induction that :

1. $(1+a)^n \geq 1 + na$ for all n in $\mathbb{N}$ and a in $\mathbb{R}^{+*}$
2. "$4^n + 2$ *is divisible by* 3" for any n in $\mathbb{N}$

3. Sequences convergence

3.1 Infinite limit

Definition :

3.2 Finite limit

3.3 Sequence without any limit

3.4 Limits by comparison

3.5 Algebra of limits

3.6 Arithmetic and geometric sequences

3.7 Bounded sequences

3.8 Monotonic sequences