TEST N°1

FRIDAY SEPTEMBER 21^{ST} 2012

EXERCICE 1. (2,5 PTS)

Opposite are the graphs of two functions, one being the derivative of the other.

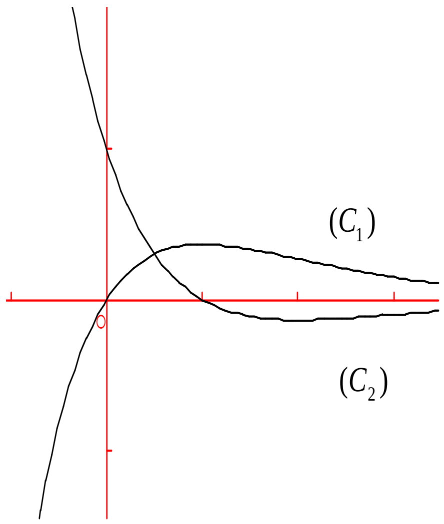
1°) Find which is C_f f and which is $C_{f'}$.

2°) Answer by True or False (no justification required)

$f(x)=0$ has two solutions

$f'(x)=0$ has no solution

$f''(x)=0$ has one solution



EXERCICE 2. (8,5 PTS)

1. Find the derivative of the following functions and briefly say how you would study their sign:

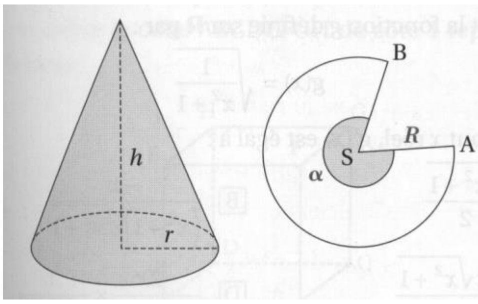
$f : x \mapsto (2x+1)^3$ $\quad$ $g : x \mapsto \dfrac{x^2}{3x-1}$ $\quad$ $h : x \mapsto x\sqrt{1-x}$

$i : x \mapsto \sqrt{3+x^2}$ $\quad$ $k : x \mapsto \dfrac{1}{\sqrt{5x-3}}$

2. Work out the equation of the tangent at -1 to the graph of f.

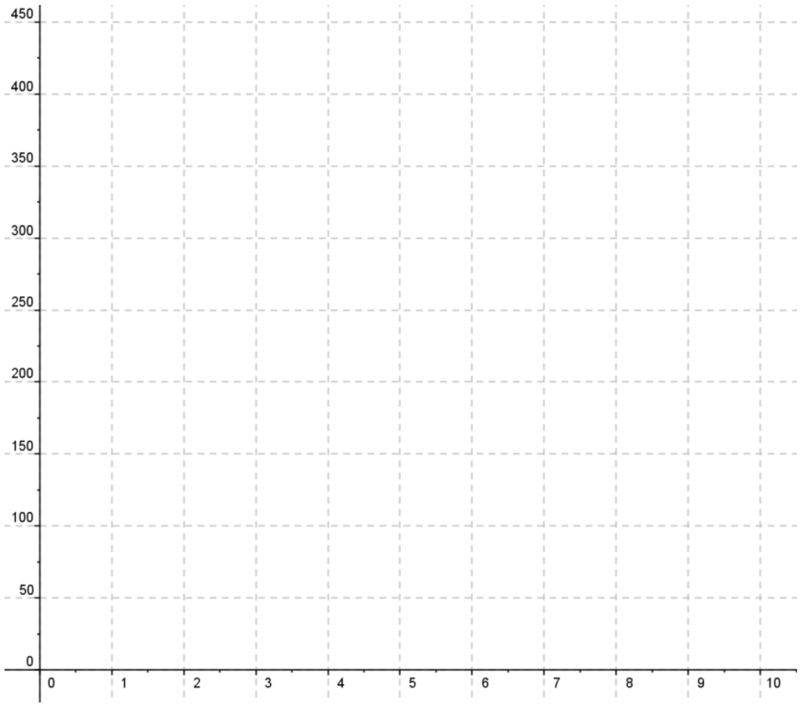
3. How would you check the differentiability of h at 1 ? (don't do it)

EXERCICE 3. (9 PTS)



A cone with height h and radius r is made from a circular sector with radius R, by joining the points A and B. The purpose of the exercise is to find the value of α such that the volume of the cone is maximal.

1. Which values of α are leading to a volume equal to 0 ?
2. Express h with regards to R and r and prove that $0 \leq r \leq R$.
3. Prove that the volume of the cone is $V = \frac{\pi}{3} r^2 \sqrt{R^2 - r^2}$.
4. Consider f the function defined on $[0; R]$ by $f(x) = \frac{\pi}{3} x^2 \sqrt{R^2 - x^2}$. Sketch up its table of variations.
5. For R=10 draw up carefully the graph of f in the coordinate system opposite.



6. Prove that f has a maximum for $x = R\sqrt{\frac{2}{3}}$. Make this value appear on your graph.
7. Prove that the volume of the cone is maximal when $\frac{r}{R} = \sqrt{\frac{2}{3}}$
8. Give an approximate value (rounded to the tenth) of α which achieves this maximum.

BONUS POINTS ON MATH CULTURE (2.5 PTS)

(on the BBC 4 podcast "Newton and Leibniz" by Markus du Sautoy)

<u>Answer with a single sentence</u>

1. How is called the new branch of maths developed by Newton and Leibniz ? What does it deal with ?
2. When did it take place ?
3. What was the problem that has occurred between Newton and Leibniz ?
4. When did Newton die and where is he buried ?

Exercice 1. (2,5 pts)

1°. f is associated with (C_1) and f ' f is associated with (C_2) because the sign of the derivative corresponds to the sense of variation of the function: (C_1) is increasing on $]-\infty;1]$ where (C_2) is above the x-axis, has a maximum when (C_2) crosses the x-axis (at 1) and is decreasing on $[1;+\infty[$ where (C_2) is under the x-axis.

2°. « $f(x)=0$ has two solutions » : False as the curve crosses the x-axis only once.

« $f'(x)=0$ has no solution » : False as the graph of f 'crosses the x-axis once.

« $f''(x)=0$ has one solution » : True since the graph of f' has a horizontal tangent

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EXERCICE 2. (8,5 PTS)

1. f is a polynomial so is differentiable at any real number and $f'(x)=3(2x+1)^2\times 2=6(2x+1)^2$

g (quotient of two polynomials) is defined and differentiable as long as its denominator is not 0, that's to say on $\left]-\infty;\frac{1}{3}\right[\cup\left]\frac{1}{3};+\infty\right[$. Then $g'(x)=\frac{2x(3x-1)-x^2(3)}{(3x-1)^2}=\frac{x(3x-2)}{(3x-1)^2}$.

h (product of two differentiable functions) is differentiable on $]-\infty;1[$ and $h'(x)=1\sqrt{1-x}+x\times\frac{1}{2\sqrt{1-x}}\times(-1)=\frac{2-3x}{2\sqrt{1-x}}$

i (composite function $\sqrt{u}$) is differentiable on $\mathbb{R}$ and $i'(x)=\frac{u'}{2\sqrt{u}}=\frac{2x}{2\sqrt{3+x^2}}=\frac{x}{\sqrt{3+x^2}}$

k (composite function $\frac{1}{u}$) is differentiable on $\left]\frac{3}{5};+\infty\right[$ and

$$k'(x)=-\frac{u'}{u^2}=-\frac{\frac{5}{2\sqrt{5x-3}}}{5x-3}=-\frac{5}{2\left(\sqrt{5x-3}\right)^{1.5}}$$

2. Work out the equation of the tangent at -1 to the graph of f.

$y=f'(-1)(x+1)+f(-1)=6(x+1)-1=6x+5$

$\boxed{y=6x+5}$ is the equation of the tangent at -1 to the graph of f

3. i is differentiable at 1 if $\frac{i(1+h)-i(1)}{h}$ has a finite limit at 0.

Exercice 3. (9 pts)

A cone with height h and radius r is made from a circular sector with radius R, by joining the points A and B. The purpose of the exercise is to find the value of α (in rad) such that the volume of the cone is maximal.

1. *Which values of α are leading to a volume equal to 0 ?*

   The volume is 0 when $\alpha = 0$ or $\alpha = 2\pi$.

2. *Express h with regards to R and r and prove that* $0 \le r \le R$.

   Thanks to Pythagoras, we have $R^2 = h^2 + r^2$, hence $h = \sqrt{R^2 - r^2}$. Since $AB \le 2\pi R$ and $AB = 2\pi r$ (since AB is the perimeter of the basis of the cone), then $r \le R$. And obviously $r \ge 0$.

3. *Prove that the volume of the cone is* $V = \frac{\pi}{3} r^2 \sqrt{R^2 - r^2}$.

   The volume of the cone is $V = \frac{1}{3} A_{basis} \times h = \frac{1}{3} \pi r^2 \sqrt{R^2 - r^2} = \frac{\pi}{3} r^2 \sqrt{R^2 - r^2}$.

4. *Consider f the function defined on* $[0; R]$ *by* $f(x) = \frac{\pi}{3} x^2 \sqrt{R^2 - x^2}$. *Sketch up its table of variations.*

   Let's study the variations of f :

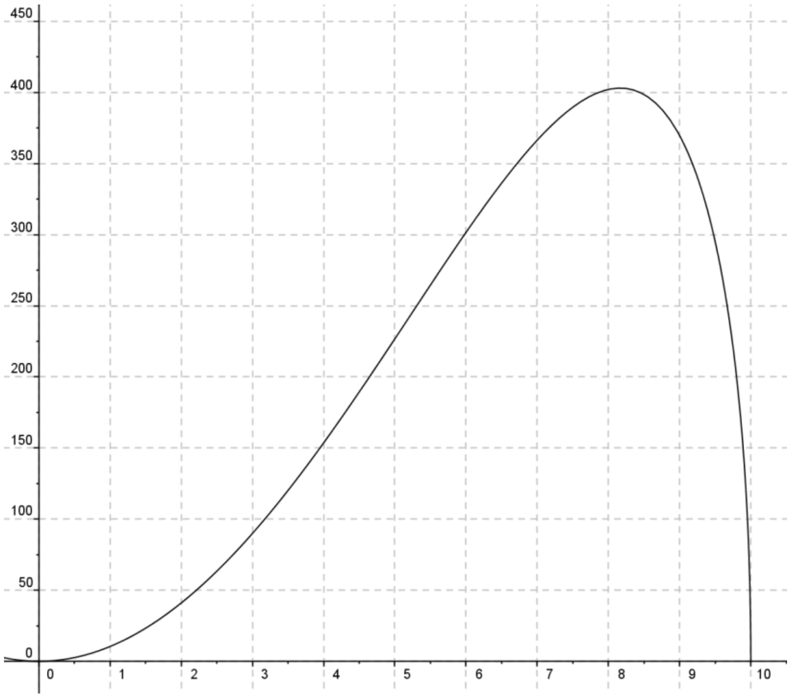
   It is differentiable on $[0; R[$ and $f = k \times (u \times v)$ with $u = x^2$ and $v = \sqrt{R^2 - x^2}$, thus $u' = 2x$ and $v' = \frac{-2x}{2\sqrt{R^2 - x^2}} = \frac{-x}{\sqrt{R^2 - x^2}}$. Then

$$f'(x) = \frac{\pi}{3} \times \left(2x\sqrt{R^2 - x^2} + x^2 \frac{-x}{\sqrt{R^2 - x^2}} \right) = \frac{\pi}{3} \times \frac{2x(R^2 - x^2) - x^3}{\sqrt{R^2 - x^2}} = \frac{\pi}{3} \times \frac{2xR^2 - 3x^3}{\sqrt{R^2 - x^2}} = \frac{\pi}{3} \times \frac{x(2R^2 - 3x^2)}{\sqrt{R^2 - x^2}}$$

   It has the same sign as $x(2R^2 - 3x^2)$. The quadratics $2R^2 - 3x^2$ is positive on $\left[0; R\sqrt{\frac{2}{3}}\right]$ (sign of a (-3) outside its roots), and hence the table of variation is :

x	0		$R\sqrt{\frac{2}{3}}$		R
x		+	0	+	
$2R^2 - 3x^2$		+	0	−	
$f'(x)$		+	0	−	
$f(x)$	0	↗	$f\left(R\sqrt{\frac{2}{3}}\right)$	↘	0

5. *For R=10 draw up carefully the graph of f in the coordinate system opposite.*



6. *Prove that f has a maximum for* $x = R\sqrt{\frac{2}{3}}$.

   For $x = R\sqrt{\frac{2}{3}}$, the derivative vanishes while changing its sign from + to -, f has thus a maimum for this value

7. Prove that the volume of the cone is maximal when $\frac{r}{R} = \sqrt{\frac{2}{3}}$

   The maximal volume is achieved when $r = R\sqrt{\frac{2}{3}}$ (roughly 8.16 on the graph above), so when $\frac{r}{R} = \sqrt{\frac{2}{3}}$

8. Give an approximate value (rounded to the tenth) of α which achieves this maximum.

   $2\pi r = AB = \alpha R$, so $\frac{r}{R} = \frac{\alpha}{2\pi}$. Since when $\frac{r}{R} = \sqrt{\frac{2}{3}}$ the volume is maximal, this value is achieved when $\frac{\alpha}{2\pi} = \sqrt{\frac{2}{3}}$, or $\alpha = 2\pi\sqrt{\frac{2}{3}} \approx 5.1\text{rad} \approx 294°$ rad

BONUS POINTS ON MATH CULTURE (2.5 PTS)

1. *How is called the new branch of maths developed by Newton and Leibniz ? What does it deal with ?*
   It is called Calculus. It deals with the study of moving objects.
2. *When did it take place ?*
   In the late XVII century
3. *What was the problem that has occurred between Newton and Leibniz ?*
   Newton claimed he was the first to have discovered the method of Calculus, although Leibniz published his workings 20 years earlier.
4. *When did Newton die and where is he buried ?*
   In died in 1727. He is buried in Westminster abbey in London.