TLE S TEST N°2

FRIDAY OCTOBER 12TH 2012

Ecole Internationale PACA à Manosque Tle S

duration : 1h30

EXERCICE 1. (12.5 PTS)

1. Simplify the following expressions : $A=\left(e^{x}\right)^{3}e^{-2x}$ $B=\dfrac{e^{3x-1}}{e^{2-x}}$

2. Solve in $\mathbb{R}$: $e^{3-x}-1=0$ $\dfrac{e^{2x-1}}{e^{3x+1}}\geq\dfrac{1}{e^{2}}$

3. Find the limits of the following function at the bounds of its domain : $f(x)=\dfrac{e^{x}+e^{-x}}{2}$ on $\mathbb{R}$

4. Work out the derivatives of the following functions on $\mathbb{R}$:
$f:x\mapsto x^{3}-e^{3}+e^{x}$ $h:x\mapsto 2e^{3x}+e^{-2x}$

5. Work out the equation of the tangent at 2 to the curve whose equation is $y=(x-1)\left(2-e^{-x}\right)$.

6. Let f be a function defined and continuous on [-3 ;4].
a) Find the number of solutions of the following equations : $a)\ f(x)=3$ $b)\ f(x)=0$.
b) Work out, depending on the value of m, the number of solution of the equation $f(x)=m$

x	-3		0		1		2		4
Variations of f	1	↗	5	↘	1	↘	-3	↗	1

EXERCICE 2. (7.5 PTS) ADAPTED FROM BAC S NOUVELLE CALEDONIE 11/2004

Let's consider the function f defined on $\mathbb{R}$ by $f(x)=\dfrac{x}{e^{x}-x}$. Let's denote (C) its graph in an orthonormal Cartesian coordinate system with unit 2 cm on the x-axis and 5 cm on the y-axis.

Part A

Let g be the function defined on $\mathbb{R}$ by $g(x)=e^{x}-x-1$.

1. Analyse the variations of g on $\mathbb{R}$. Deduce then the sign of $g(x)$.
2. Justify that, for any x, $\left(e^{x}-x\right)$ is strictly positive.

Part B

1. a. Calculate $f'(x)$, f' being the derivative of f.
b. Analyse the variations of f and draw up its table of variations.
2. a) Prove that the equation $2f(x)=1$ has exactly 2 solutions on $\mathbb{R}$.
b) Give their approximate values to the thousandth.
3. a. Work out the equation of the tangent (T) to the curve (C) at the point where x = 0.
b. Using part A, analyse the position of (C) compared with (T).
c. Draw up (T) and the curve (C). (units : 2cm on the x-axis, 5cm on the y-axis)

BONUS POINTS ON MATH CULTURE (3 PTS)

(on the Clay Institute public lecture video "A tribute to Euler" by Wiliam Dunham)

Answer with a single sentence

1. When did Euler live ? In which country was he born ?
2. A had a close friendship with another famous mathematician. Which one ?
3. How old was he when he first graduated ?
4. Where did he spend most of his career ?
5. What happened to him in 1770? What was the effect on his work ?
6. Quote what is ,according to you, the most impressive Euler's achievement.

Exercice 1. (12.5 pts)

1. $A=\left(e^x\right)^3 e^{-2x}=e^{3x}e^{-2x}=e^x$ $\qquad B=\dfrac{e^{3x-1}}{e^{2-x}}=e^{3x-1-(2-x)}=e^{4x-3}$

2. $e^{3-x}-1=0 \Leftrightarrow 3-x=0$ because the exp function is strictly monotonic on $\mathbb{R}$ et and $e^0=1$. Hence $S=\{3\}$.

$\dfrac{e^{2x-1}}{e^{3x+1}}\geq\dfrac{1}{e^2}\Leftrightarrow e^{-x-2}\geq e^{-2}\Leftrightarrow -x-2\geq -2$ because the exp function is strictly monotonic on $\mathbb{R}$. $S=\left]-\infty;0\right]$.

3. $\lim\limits_{x\to+\infty} e^x=+\infty$ et $\lim\limits_{x\to+\infty} e^{-x}=0$, hence, by sum, $\lim\limits_{x\to+\infty}\dfrac{e^x+e^{-x}}{2}=+\infty$. The same way $\lim\limits_{x\to-\infty}\dfrac{e^x+e^{-x}}{2}=+\infty$

4. $f: x\mapsto x^3-e^3+e^x$ is differentiable on $\mathbb{R}$ and $f'(x)=3x^2+e^x$. $h: x\mapsto 2e^{3x}+e^{-2x}$ is differentiable on $\mathbb{R}$ and $h(x)=6e^{3x}-2e^{-2x}$ (form e^u).

5. Let's denote $f(x)=(x-1)\left(2-e^{-x}\right)$.

Then $f'(x)=u'v+uv'=\left(2-e^{-x}\right)+(x-1)e^{-x}=2-e^{-x}+xe^{-x}-e^{-x}=xe^{-x}+2\left(1-e^{-x}\right)$. $f'(2)=2e^{-2}+2\left(1-e^{-2}\right)=2$ and $f(2)=2-e^{-2}$. The tangent to the curve whose equation is $y=(x-1)\left(2-e^{-x}\right)$ at 2 has then the equation $y=f'(2)(x-2)+f(2)=2(x-2)+2-e^{-2}=2x-2-e^{-2}$.

6. Done in class

Exercice 2. (10 pts) Bac S Nouvelle Calédonie Novembre 2004

Part A

1. g is differentiable on $\mathbb{R}$ and $g'(x)=e^x-1$. $g'(x)>0\Leftrightarrow e^x>1\Leftrightarrow x>0$ (since the exp function is strictly increasing on $\mathbb{R}$.). $g'(0)=0$, we thus have the following table of variations :

Since $g(0)=0$, 0 is the absolute minimum of g and $g(x)\geq 0$ for any real number x.

2. $g(x)\geq 0\Leftrightarrow e^x-x\geq 1$ thus, for any x, $\left(e^x-x\right)$ is strictly positive.

x	$-\infty$	0	$+\infty$
Sign of g '	$-$	0	$+$
Variation of g	↘	0	↗
Sign of g	$+$		$+$

Part B

1. a. $\left(e^x-x\right)$ being strictly positive, f can be differentiated on $\mathbb{R}$. f is in the form $\dfrac{u}{v}$.

$f'(x)=\dfrac{e^x-x-x\left(e^x-1\right)}{\left(e^x-x\right)^2}=\dfrac{e^x(1-x)}{\left(e^x-x\right)^2}$.

b. $\left(e^x-x\right)^2>0$ and $e^x>0$, thus f' has the same sign as $1-x$.

x	$-\infty$	α_1	1	α_2	$+\infty$
Sign of f '		$+$	0	$-$	
Variation of f	-1 ↗	0.5	$\dfrac{1}{e-1}>0.5$	0.5	↘ 0

2. a) The equation $2f(x)=1$ is

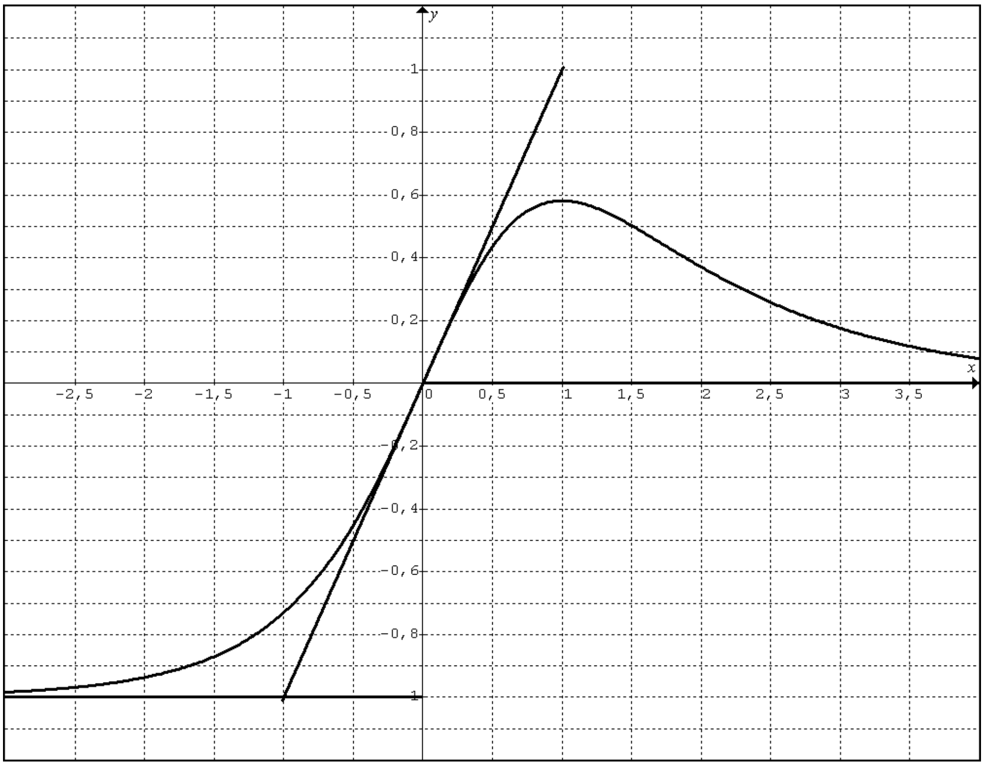
equivalent to $f(x)=0.5$. According to the table of variation and the IVT, it has exactly 2 solutions on $\mathbb{R}$, one on $]-\infty;1[$, and one on $]1;+\infty[$.

b) $\alpha_1 \approx 0.619$ and $\alpha_2 \approx 1.512$.

3. a. The tangent (T) to the graph (C) at 0 has for equation $y = f'(0)(x-0)+f(0)=x$ since $f'(0)=f(0)=0$.

b. Let's consider $\varepsilon(x)=f(x)-x=\frac{x}{e^x-x}-\frac{x(e^x-x)}{e^x-x}=\frac{x(1-e^x+x)}{e^x-x}=-\frac{xg(x)}{e^x-x}$. According to part A , $g(x)\geq 0$ and $(e^x-x)>0$ for any x, $\varepsilon(x)$ has thus the same sign as –x , that's negative on $\mathbb{R}^+$ and positive on $\mathbb{R}^-$. Hence (C) is above (T) on $\mathbb{R}^-$ and underneath on $\mathbb{R}^+$.

c.



BONUS POINTS ON MATH CULTURE (3 PTS)

(on the Clay public lecture video "A tribute to Euler" by Wiliam Dunham)

1. 18^{th} century. He was born in Switzerland ?
2. Johann Bernoulli.
3. He was 15 when he first graduated.
4. He spent most of his career in the science academy in St Petersburgh.
5. In 1770, he had a surgery that went wrong and then almost became blind. His production was still higher.
6. One of the most impressive Euler's achievement is the incredible amount of papers he produced, still not yet completely published.