Sequences part I
Exercises sheet

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<u>Exercise 0</u> Write a program for your calculator which calculate the speed of convergence of a sequence define by recurrence :

1. Input : The limit l , the first term u and the accuracy p we want. (the recurrence function is in Y1)
2. Output : The first rank n for which $u_n \in]\ell - p ; \ell + p[$.
3. Change the program to do the same thing if the sequence is define by Y1(n)

<u>Exercise 1</u> For each sequence:

1. Study the variations.
2. Prove if it has an upper or lower bound (you can use a graph).
3. Study its limits.
4. When the sequence converges to ℓ , find the rank *p* from which all the terms of the sequence (u_n) are in the interval $\left]\ell - 10^{-4} ; \ell + 10^{-4}\right[$.

a) $u_n = \frac{n^2}{n^2+1}$, $n \in \mathbb{N}$	b) $u_n = \frac{1}{\sqrt{n}+1}$, $n \in \mathbb{N}$	c) $u_n = 1+2+3+...+n$, $n \in \mathbb{N}$
d) $u_n = \frac{1}{1\times 2} + \frac{1}{2\times 3} + ... + \frac{1}{n(n+1)}$, $n \geq 1$ We will prove: $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$	e) $n \in \mathbb{N}$, $\begin{cases} u_0 = 2 \\ u_{n+1} - u_n = \sqrt{2} \end{cases}$	f) $n \in \mathbb{N}$, $\begin{cases} u_0 = 2 \\ u_{n+1} - u_n = u_n\sqrt{2} \end{cases}$

<u>Exercise 2</u> Let (u_n) be a sequence defined for all $n \in \mathbb{N}$ by : $\begin{cases} u_0 = 5 \\ u_{n+1} = \frac{1}{2}u_n + 3 \end{cases}$

1. Is (u_n) a geometric progression?
2. Prove that the sequence defined by $v_n = u_n - 6$ is a geometric progression.
3. Express v_n then u_n using (in terms of) *n*.
4. Study (u_n) (variations, bounds, convergence).
5. Find the sum of the first 10 terms of (u_n).

<u>Exercise 3</u> For each sequence, calculate u_1, u_2, u_3, u_4 and check your results using your calculator.

a) $\begin{cases} u_0 = 0 \\ u_{n+1} = \sqrt{u_n + 6} \end{cases}$ b) $\begin{cases} u_0 = -2 \\ u_{n+1} = \sqrt{u_n + 6} \end{cases}$ c) $\begin{cases} u_1 = 1 \\ u_{n+1} = \frac{u_n}{u_n + 1} \end{cases}$ d) $\begin{cases} u_0 = 1 \\ u_{n+1} = u_n - \frac{1}{n+1} \end{cases}$

<u>Exercise 4</u> Study the variations of the following sequences:

a) $u_n = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{n}}$, $n \geq 1$ b) $u_n = 3n + (-1)^n$, $n \in \mathbb{N}$ c) $u_n = n - 3^n$, $n \in \mathbb{N}$ d) $u_n = \frac{n}{3^n}$, $n \in \mathbb{N}$

d) $u_n = \frac{2^n + 1}{3^n}$, $n \in \mathbb{N}$ e) $u_n = \frac{1}{2} \times \frac{3}{4} \times ... \times \frac{2n-1}{2n}$, $n \geq 1$ f) $\begin{cases} u_0 = 1 \\ u_{n+1} = u_n + 2n + 3 \end{cases}$ g) $\begin{cases} u_0 = 0,5 \\ u_{n+1} = -u_n^2 + 3u_n - 1 \end{cases}$

<u>Exercise 5</u> Find the bounds of the following sequences:

a) $u_n = \frac{3n^2}{n^2+1}$, $n \in \mathbb{N}$ b) $u_n = \sqrt{2 + \cos n}$, $n \in \mathbb{N}$ c) $u_n = \left(\frac{1}{4}\right)^n - 1$, $n \in \mathbb{N}$ d) $u_n = 1 + \frac{2}{7} + ... + \left(\frac{2}{7}\right)^n$, $n \in \mathbb{N}$

Exercise 6 Find the limits of the following sequences:

a) $u_n = 3n^2 - n + \frac{1}{n}$, $n \geq 1$ b) $u_n = \frac{n}{2n^2+1}$, $n \in \mathbb{N}$ c) $u_n = \frac{1}{5} + \frac{1}{5^2} + \ldots + \frac{1}{5^n}$, $n \in \mathbb{N}$

d) $u_n = \frac{4}{3-2^n}$, $n \in \mathbb{N}$ e) $u_n = \sqrt{n+1} - \sqrt{n}$, $n \in \mathbb{N}$ f) $u_n = \frac{3^n + 4^n}{3^n + 2^n}$, $n \in \mathbb{N}$

g) $u_n = 2n + (-1)^n$, $n \in \mathbb{N}$ f) $u_n = \left(\frac{3}{4}\right)^n \cos n$, $n \in \mathbb{N}$ h) $u_n = \frac{3 - \sin n}{n}$, $n \geq 1$

Exercise 7 Let (u_n) be a sequence defined by $u_n = 3 + \frac{\sqrt{n}}{n+(-1)^n}$, $n \geq 2$. Prove that $\frac{\sqrt{n}}{n+1} \leq u_n - 3 \leq \frac{\sqrt{n}}{n-1}$.

Deduce the limit of (u_n).

Exercise 8 Prove that if a sequence converges then it is bounded.

Exercise 9 For each sequence, find the explicit expression of u_n and prove it, using induction:

a) $\begin{cases} u_0 = 0 \\ u_{n+1} = u_n + n \end{cases}$ b) $\begin{cases} u_1 = 1 \\ u_{n+1} = \frac{u_n}{u_n + 1} \end{cases}$

Exercise 10 Using induction, find the variation of the following sequences :

a) $\begin{cases} u_0 = 5 \\ u_{n+1} = \sqrt{u_n + 2} \end{cases}$ b) $\begin{cases} u_0 = 2 \\ u_{n+1} = \frac{u_n}{u_n^{\ 2} + 1} \end{cases}$

Exercise 11 Let (u_n) be the sequence defined for all $n \in \mathbb{N}$ by $\begin{cases} u_0 = 3 \\ u_{n+1} = 2 + \frac{1}{u_n} \end{cases}$. Prove that $2 \leq u_n \leq 3$.

Exercise 12: Let (u_n) be the sequence defined for all $n \in \mathbb{N}$ by $\begin{cases} u_0 = 1 \\ u_{n+1} = \frac{1}{4}u_n + 3 \end{cases}$

1) Using your calculator and a graph, put forward an hypothesis about the convergence of (u_n).
2) Let (v_n) be the sequence defined for all $n \in \mathbb{N}$ by $v_n = u_n - 4$.
   a) For all natural numbers n, express v_{n+1} thanks to v_n. What kind of sequence is (v_n)?
   b) Prove that $u_n = -5 \times (\frac{1}{3})^n + 6$ for all natural numbers n.
   c) Prove that (u_n) converges and find its limit.

Exercise 13 Let (u_n) be the sequence defined for all $n \in \mathbb{N}$ by: $u_n = 2^n - n$

1) Using your calculator, put forward an hypothesis about the convergence of (u_n).
2) Study the variations of (u_n).
3) Prove that, for all natural number n, $n \geq 3$, $u_n \geq 1{,}5^n$. Deduce the behaviour of (u_n) at infinity.

Exercise 17

Part A: Let (u_n) and (S_n) be two sequences defined on $\mathbb{N}$ by:

$\begin{cases} u_0 = 13 \\ u_{n+1} = \frac{1}{5}u_n + \frac{4}{5} \end{cases}$ $S_n = \sum_{k=0}^{n} u_k = u_0 + u_1 + \ldots\ldots\ldots + u_n$

1) Using induction, prove that $u_n = 1 + \frac{12}{5^n}$ for all natural number n. Deduce the limit of (u_n).

2) a) Find the variation of (S_n).

- Express S_n as a function of n.
- Find the limit of (S_n).

Part B: Let (x_n) be a sequence defined on $\mathbb{N}$ and (S_n) be a sequence defined on $\mathbb{N}$ by: $S_n = \sum_{k=0}^{n} u_k$.

Justify if each proposition is true or false:

- If (x_n) converges, then (S_n) converges too.
- The sequences (x_n) and (S_n) have the same variation.

Exercise 18

In a shooting stand, a person shoot several times to hit several targets. The probability of the first target being hit is $\frac{1}{2}$. When a target is hit, the probability if the following target being hit is $\frac{3}{4}$. When a target is not hit, the probability if the following target being hit is $\frac{1}{2}$.

For all natural numbers n, $n > 0$:

- A_n is the event: «the n[th] target is hit»;
- $\overline{A_n}$ is the event: «the n[th] target is not hit»;
- a_n is the probability of A_n
- b_n is the probability of $\overline{A_n}$.

1) Using a probability tree diagram, find a_1, a_2, b_1, b_2.
2) Prove that $a_{n+1} = \frac{3}{4}a_n + \frac{1}{2}b_n$ and then, that $a_{n+1} = \frac{1}{4}a_n + \frac{1}{2}$.
3) Let (U_n) be the sequence defined by $U_n = a_n - \frac{2}{3}$.

a) Prove that (U_n) is a geometric progression. Give the first term and the common ratio of (U_n).
b) Express U_n as a function of n, then express a_n as a function of n.
c) Find the limit of the sequence (a_n).
d) Find the smaller natural number n so that $a_n \geq 0.6665$.

Exercise 19 Calculator program to calculate the terms of u_0 *given and* $u_{n+1} = f(u_n)$

Nom= SuiteRec

To type	*What is the calculator doing*	*Where to find what we need*
Prompt N,U	These are the values you will be asked to run the program. U and N are the first term of (u_n) and its rank.	In ***PRGM E/S***: Prompt, Disp, Lbl, Goto, Pause
Lbl 0	**Lbl** labels a line of the program. It needs to be numbered. Here, it is line 0.	
N+1 $\rightarrow$ N Y1(U) $\rightarrow$ U	We calculate the next term and its rank and we display them on the screen.	$\rightarrow$: is "***sto*** "
Disp «N»,N Disp «U»,U		
Pause		In ***VAR Y-VAR f unction*** : Y1
Goto 0	To **run the program**, remember to enter the function f in «*f(x)*», Y1.	

<u>**Exercise 20:**</u>

1) We suppose the number $4^p + 1$, with p a random natural number, can be divided by 3. Prove that $4^{p+1} + 1$ can be divided by 3.
2) Can we conclude that $4^p + 1$ can be divided by 3 for all natural numbers n?

3) Prove, using induction, that $4^n - 1$ can be divided by 3 for all natural numbers n.

Exercise 21 :

1) Let (u_n) be a sequence defined by: $u_0 = 5$ *and* $u_{n+1} = 3\,u_n,\ n \in \mathbb{N}$. Prove that $u_n = 5 \times (3)^n$, for all natural numbers n.
2) Let (u_n) be a sequence defined by: $u_0 = 0$ *and* $u_{n+1} = 2u_n + 1,\ n \in \mathbb{N}$. Prove that $u_n = 2^n - 1$ for all natural numbers n.

Exercise 22 Let (u_n) be a sequence defined by $u_0 = 0$ *and* $u_{n+1} = \sqrt{u_n + 6},\ n \in \mathbb{N}$. Using induction, prove that, for all natural numbers n, $0 \le u_n \le 3$.

Exercise 23 : 1) Prove that, for all natural numbers n, $1 + 2 + ... + n = \dfrac{n(n+1)}{2}$.

2) Prove that, for all natural numbers n, $1^2 + 2^2 + ... + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

Exercise 24 :

To play a lottery game, you are asked to turn two wheels of 20 sectors.
The wheel A has 18 black sectors and 2 red sectors.
The wheel B has 16 black sectors and 4 red sectors.
The arrows of each wheels point to each sector with the same probability.

The rule of the lottery game is:

- The player pays 1euros and turns wheel A.
- If the arrow points to a red sector, then the player turns wheel B. He write down the colour of the sector the arrow points to, and the game is over.
- If the arrow points to a black sector, then the player turns wheel A again. He write down the colour of the sector the arrow points to, and the game is over.

1) Draw up the probability tree diagram of this random experiment.
2) Let E and F be the following events:
   E: «At the end of the game, the two sectors are red»
   F: «At the end of the game, only one sector is red».

Prove that $p(E) = 0{,}02$ and $p(F) = 0{,}17$.

3) If the two sectors are red, the player wins 10 euros; if only one sector is red, the player wins 2 euros; otherwise, he doesn't win anything.

X is the random variable equal to the amount of money in euros won or lost by the player.

   a) Write the probability distribution of X.
   b) Find the mean value of X and comment your result.

4) The player wants to play n games (n is a natural number greater or equal to 2).

Each game is independent from the other.

a. Prove that the probability that the player turns wheel B at least once is equal to: $p_n = 1 - (0.9)^n$.
b. Justify that the sequence (p_n) converges and find its limit.
c. What is the smallest value of n so that $p_n > 0.9$?

Sequences part I
Exercises sheet answers (samples)

Exercise 0

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PROGRAM:SEQSPEED
:Prompt L
:Prompt P
:Prompt U
:0→N
:U→X
:While abs(Y1(X)
-L)≥P
:While abs(Y1(X)
-L)≥P
:N+1→N
:Y1(X)→X
:End
:Disp N
:
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Exercise 1 For $u_n = \dfrac{n^2}{n^2+1}$:

5. Study the variations: $u_{n+1} - u_n = \dfrac{(n+1)^2}{(n+1)^2+1} - \dfrac{n^2}{n^2+1} = ... = \dfrac{2n+1}{\left[(n+1)^2+1\right]\left[n^2+1\right]} > 0$ so u is increasing.
6. Prove if it has an upper or lower bound (you can use a graph): $u_n = \dfrac{n^2}{n^2+1} < 1$ since $n^2 < n^2+1$ and $u_n = \dfrac{n^2}{n^2+1} \geq 0$. Therefore 0 is a lower bound and 1 an upper bound.
7. Study its limits : $\lim\limits_{n\to+\infty} \dfrac{n^2}{n^2+1} = 1$ rule of the term of highest degree for polynomials
8. When the sequence converges to ℓ, find the rank p from which all the terms of the sequence (u_n) are in the interval $\left]\ell - 10^{-4}\,;\,\ell + 10^{-4}\right[$: $u_n - 1 = \dfrac{n^2}{n^2+1} - 1 = \dfrac{-1}{n^2+1}$, we have to choose n so that $n^2+1 > 10^4$ so n=100

Exercise 2 Let (u_n) a sequence defined for all $n \in \mathbb{N}$ by : $\begin{cases} u_0 = 5 \\ u_{n+1} = \dfrac{1}{2}u_n + 3 \end{cases}$

6. Is (u_n) a geometric progression? No, since the recurrence function is not in the form $u_{n+1} = b \times u_n$
7. Prove that the sequence defined by $v_n = u_n - 6$ is a geometric progression :
   $\dfrac{v_{n+1} = u_n - 6}{v_n} = \dfrac{u_{n+1} - 6}{u_n - 6} = \dfrac{0.5u_n + 3 - 6}{u_n - 6} = \dfrac{0.5u_n - 3}{u_n - 6} = \dfrac{0.5(u_n - 6)}{u_n - 6} = 0.5$ so v_n is a geometric progression with common ratio 0.5
8. Express v_n then u_n using (in terms of) n : $v_n = v_0 0.5^n = -0.5^n$ so $u_n = v_n + 6 = -0.5^n + 6$
9. Study (u_n) (variations, bounds, convergence) : the sequence 0.5^n is decreasing, so (u_n) is increasing. Its limit is 6 since -0.5^n tends to 0. It's then bounded by 5 and 6.
10. Find the sum of the first 10 terms of (u_n) : $\sum_0^9 v_n = v_0 \dfrac{1 - 0.5^{10}}{1 - 0.5} = -\dfrac{1023}{2048}$ so
   $\sum_0^9 u_n = \sum_0^9 (v_n + 6) = \sum_0^9 v_n + \sum_0^9 6 = -\dfrac{1023}{2048} + 60 = \dfrac{121857}{2048}$

Exercise 3 For a) $u_0 = 0, u_1 = \sqrt{6} \approx 2.45, u_2 = \sqrt{\sqrt{6}+6} \approx 2.91, u_3 = \sqrt{\sqrt{\sqrt{6}+6}+6} \approx 2.98, u_4 = \sqrt{\sqrt{\sqrt{\sqrt{6}+6}+6}+6} \approx 2.997$
Check with wolframalpha (u(n+1) = sqrt(u(n)+6),u(0) = 0)

Exercise 4 Study the variations of the following sequences:

For a) $u_{n+1} - u_n = \dfrac{1}{\sqrt{1}} + \dfrac{1}{\sqrt{2}} + ... + \dfrac{1}{\sqrt{n}} + \dfrac{1}{\sqrt{n+1}} - \left(\dfrac{1}{\sqrt{1}} + \dfrac{1}{\sqrt{2}} + ... + \dfrac{1}{\sqrt{n}}\right) = \dfrac{1}{\sqrt{n+1}} > 0$ so u is increasing

Exercise 5 Find the bounds of the following sequences:

For a) $u_n = \frac{3n^2}{n^2+1}$, $\frac{3n^2}{n^2+1} < 3$ since $n^2 < n^2+1$ and $\frac{3n^2}{n^2+1} \geq 0$. Therefore 0 is a lower bound and 1 an upper bound

Exercise 6 Find the limits of the following sequences:

For a) $u_n = 3n^2 - n + \frac{1}{n} = 3n^2\left(1 - \frac{1}{3n} + \frac{1}{3n^3}\right)$, $\lim 3n^2 = +\infty$ and $\lim\left(1 - \frac{1}{3n} + \frac{1}{3n^3}\right) = 1$ so, by product $\lim u_n = +\infty$.

b) $u_n = \frac{n}{2n^2+1}$, $\lim(u_n) = 0$ c) $u_n = \frac{1}{5} + \frac{1}{5^2} + \ldots + \frac{1}{5^n}$, $\lim(u_n) = \frac{1}{4}$

d) $u_n = \frac{4}{3-2^n}$, $\lim(u_n) = 0$ e) $u_n = \sqrt{n+1} - \sqrt{n}$, $\lim(u_n) = 0$ f) $u_n = \frac{3^n + 4^n}{3^n + 2^n}$, $\lim(u_n) = +\infty$

g) $u_n = 2n + (-1)^n$, $\lim(u_n) = +\infty$ f) $\lim(u_n) = 0$, $n \in \mathbb{N}$ h) $u_n = \frac{3 - \sin n}{n}$, $\lim(u_n) = 0$

Exercise 7 Let (u_n) a sequence defined by $u_n = 3 + \frac{\sqrt{n}}{n + (-1)^n}$, $n \geq 2$. Prove that $\frac{\sqrt{n}}{n+1} \leq u_n - 3 \leq \frac{\sqrt{n}}{n-1}$.

Deduce the limit of (u_n) : $\lim(u_n) = 3$

Exercise 8 Prove that if a sequence converges then it is bounded.

Exercise 9 For each sequence, find the explicit expression of u_n and prove it, using induction:

a) $\begin{cases} u_0 = 0 \\ u_{n+1} = u_n + n \end{cases}$ $u_n = 0.5n(n-1)$ b) $\begin{cases} u_1 = 1 \\ u_{n+1} = \frac{u_n}{u_n + 1} \end{cases}$ $u_n = \frac{1}{n+1}$

Exercise 10 Using induction, find the variation of the following sequences :

a) $\begin{cases} u_0 = 5 \\ u_{n+1} = \sqrt{u_n + 2} \end{cases}$ Decreasing, limit 2 b) $\begin{cases} u_0 = 2 \\ u_{n+1} = \frac{u_n}{u_n^2 + 1} \end{cases}$ Decreasing, limit 0